MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

# LECTURE #9

LAST TIME

- FUNCTORS
- BIFUNCTORS & MULTIFUNCTORS
- NATURAL TRANSFORMATIONS
- COMPOSITIONS OF NATURAL TRANSFORMATIONS

## TOPICS:

I. ISOMORPHISM OF CATEGORIES (§2.4.1)

II. EQUIVALENCE OF CATEGORIES (§§2.4.2–2.4.3)

III. MORITA EQUIVALENCE (§2.4.3)

## ≡ RECALL ≡

A **CATEGORY** $\mathscr{C}$

CONSISTS OF:

(a) **OBJECTS**.

(b) **MORPHISMS** $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \longrightarrow X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

**ASSOCIATIVITY**

$(hg)f = h(gf)$

**UNITALITY**

$id_X f = f, \; g\, id_X = g$

## ≡ RECALL ≡

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X,Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

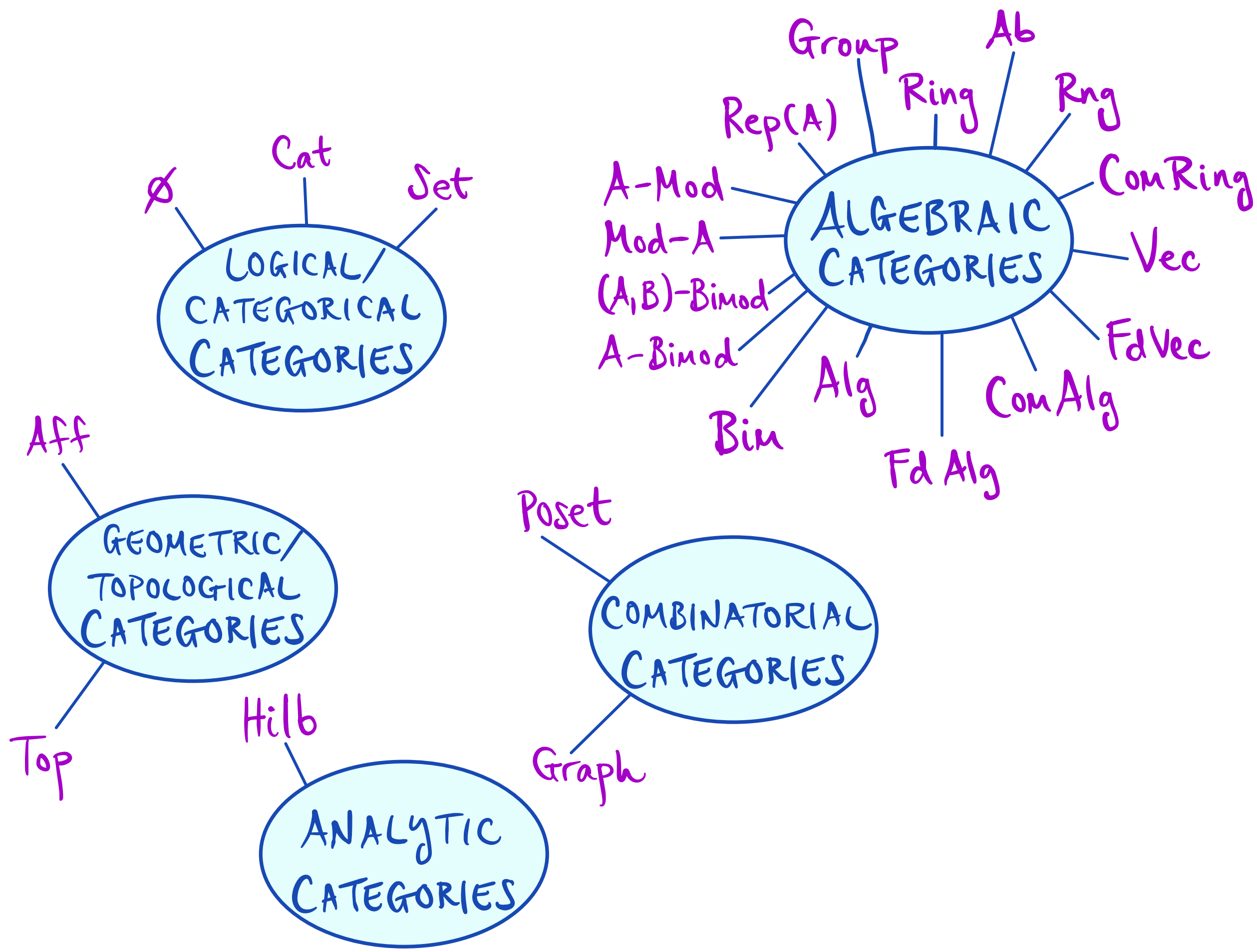
SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g\, id_X = g$



≡ RECALL ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

A CATEGORY $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

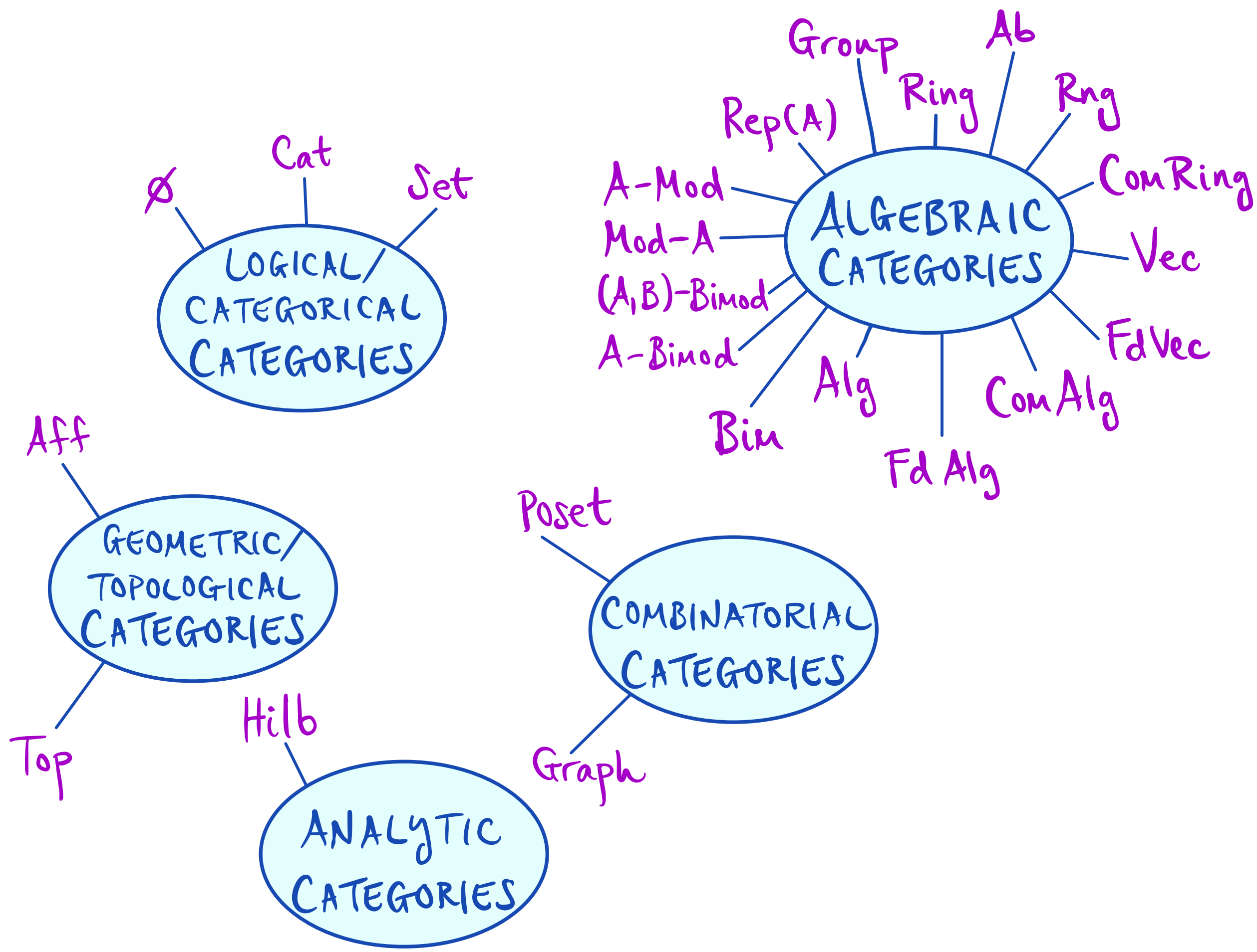
SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g\, id_X = g$



## ≡ RECALL ≡

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X,Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

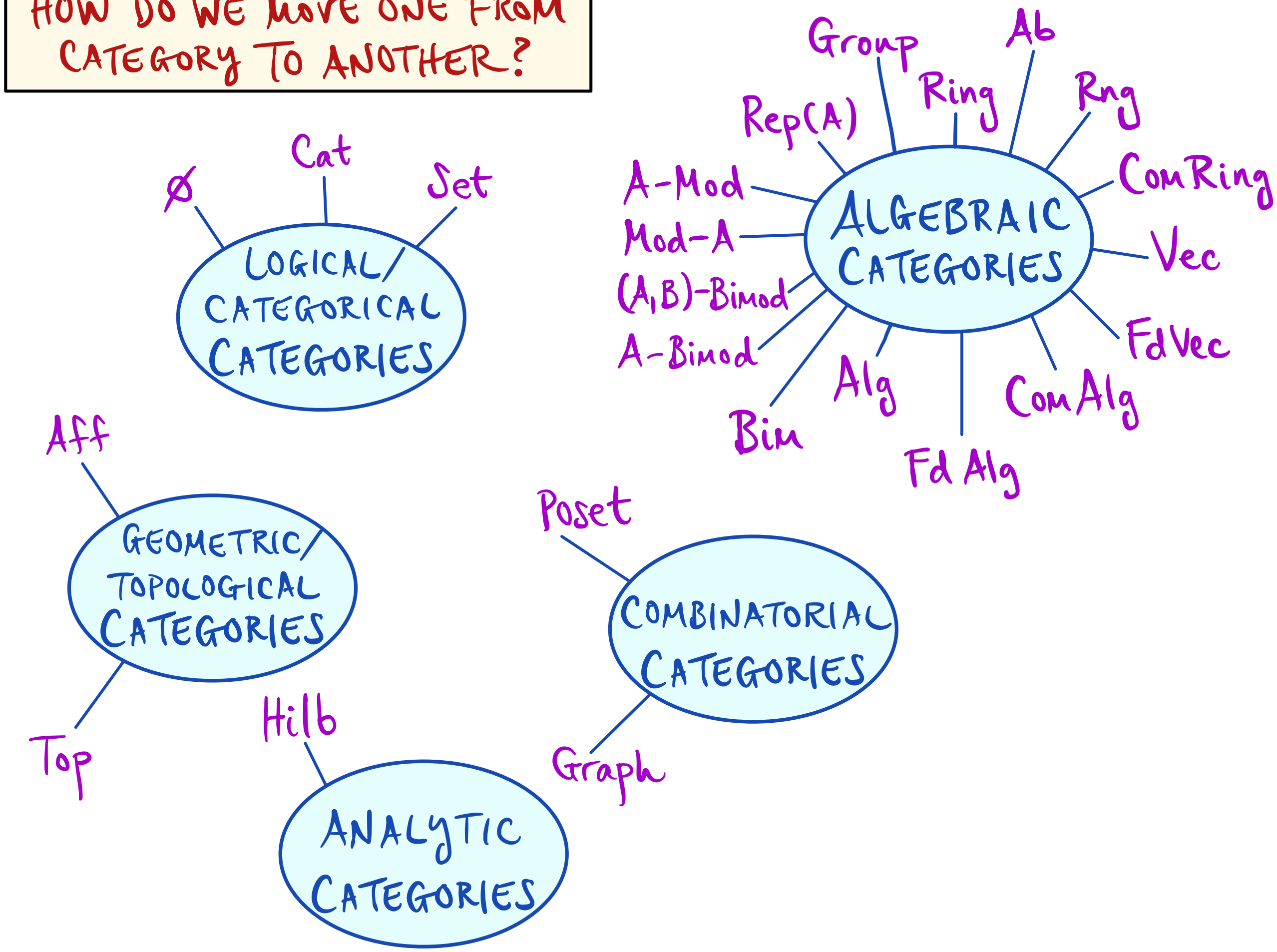
$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g\, id_X = g$

## WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?



## ≡ RECALL ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

(RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

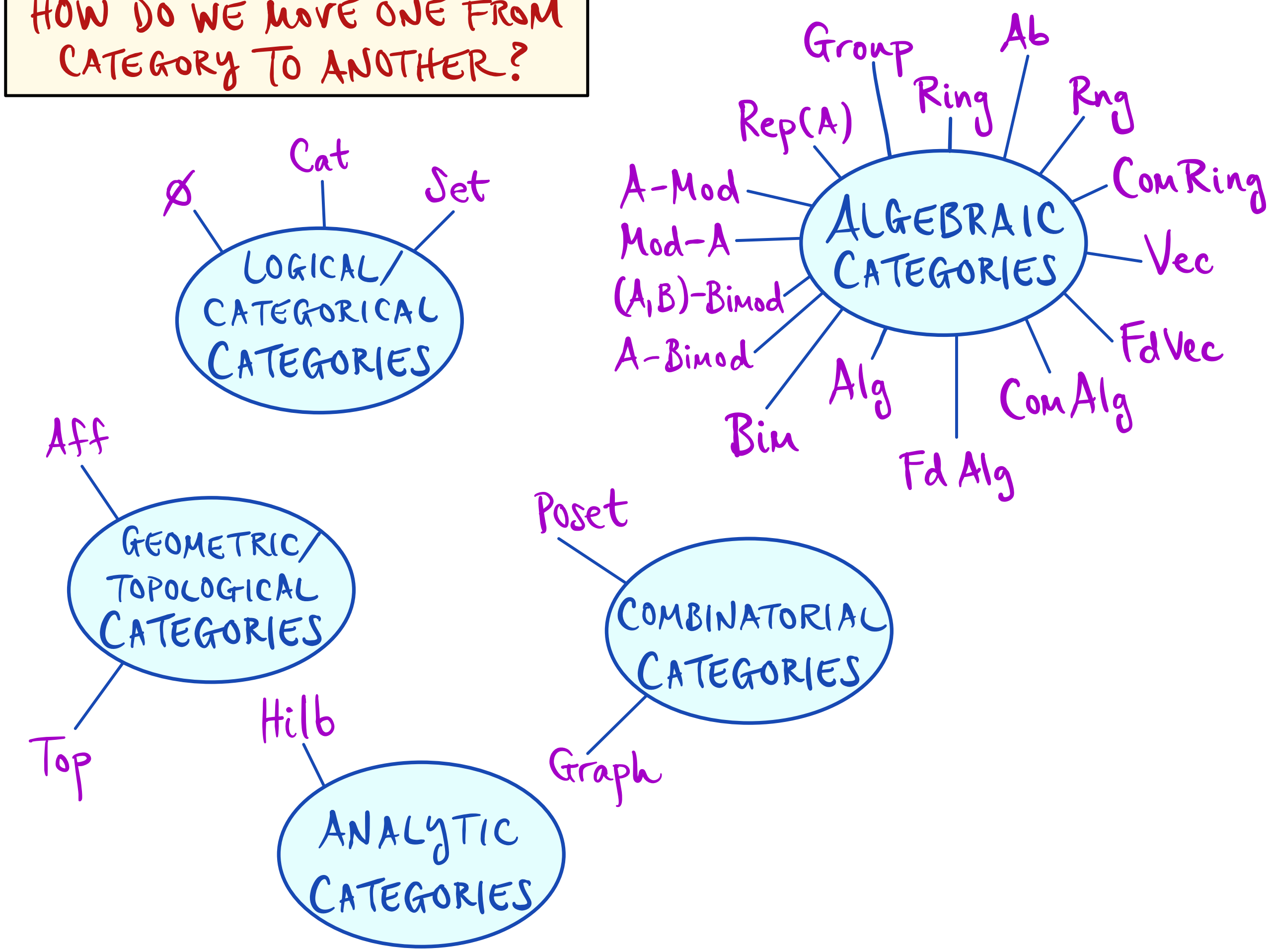
(b) $F(g): F(X) \to F(Y) \in \mathcal{D}$

(RESP., $F(g): F(Y) \to F(X) \in \mathcal{D}$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

IDENTITY & COMPOSED MORPHISMS



## ≡ RECALL ≡

**WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?**

**HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?**

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

(RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

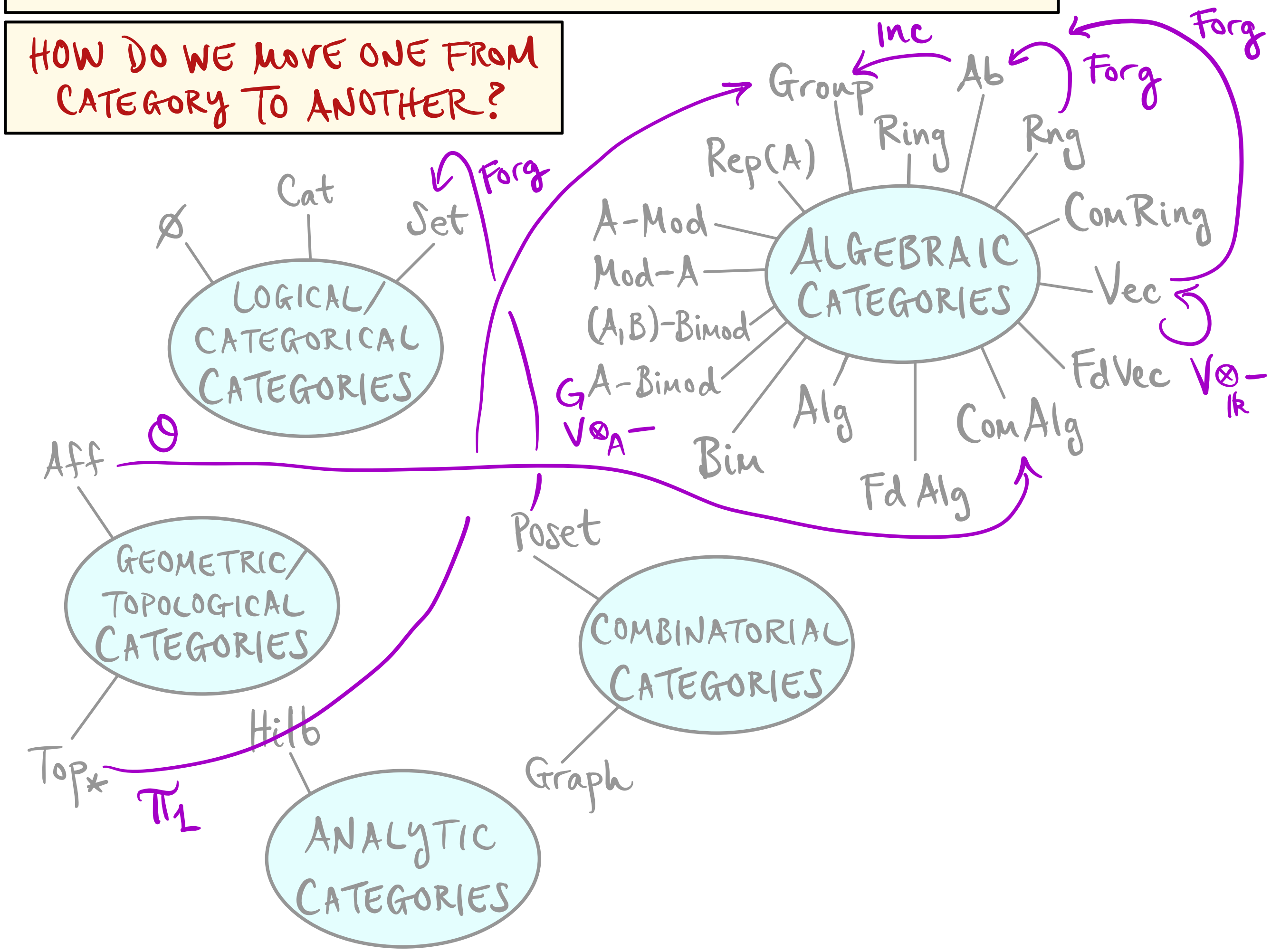
(b) $F(g) : F(X) \to F(Y) \in \mathcal{D}$

(RESP., $F(g) : F(Y) \to F(X) \in \mathcal{D}$)

$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

IDENTITY & COMPOSED MORPHISMS



# ≡ RECALL ≡

**WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?**

**HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?**

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT)

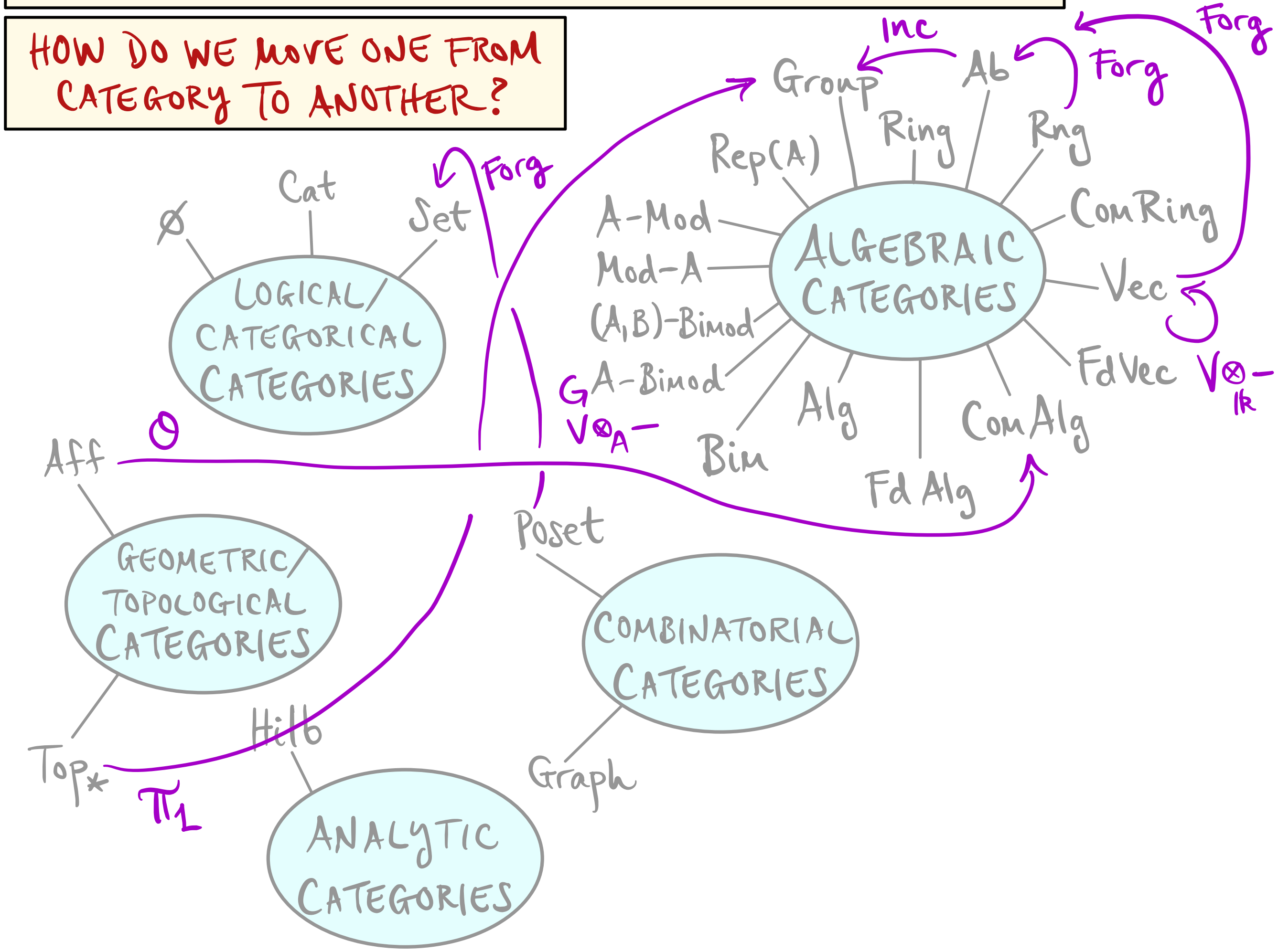
CONSISTS OF:

(a) $F(X)\in\mathcal{D}\ \forall X\in\mathcal{C}$.

(b) $F(g): F(X)\to F(Y)\in\mathcal{D}$ (RESP., $F(g): F(Y)\to F(X)\in\mathcal{D}$) $\forall g: X\to Y\in\mathcal{C}$.

RESPECTING:

IDENTITY & COMPOSED MORPHISMS




| | | | |
|---|---|---|---|
| $F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g\mapsto F(g)$ | | | F EMBEDDING: F FAITH & INJ ON OBJS |
| F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$ | F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$ | F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$ | F ESS. SURJ: $\forall Y\in\mathcal{D}, \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$ |

## ≡ RECALL ≡

**WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?**

**HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?**

A **FUNCTOR** $F : \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

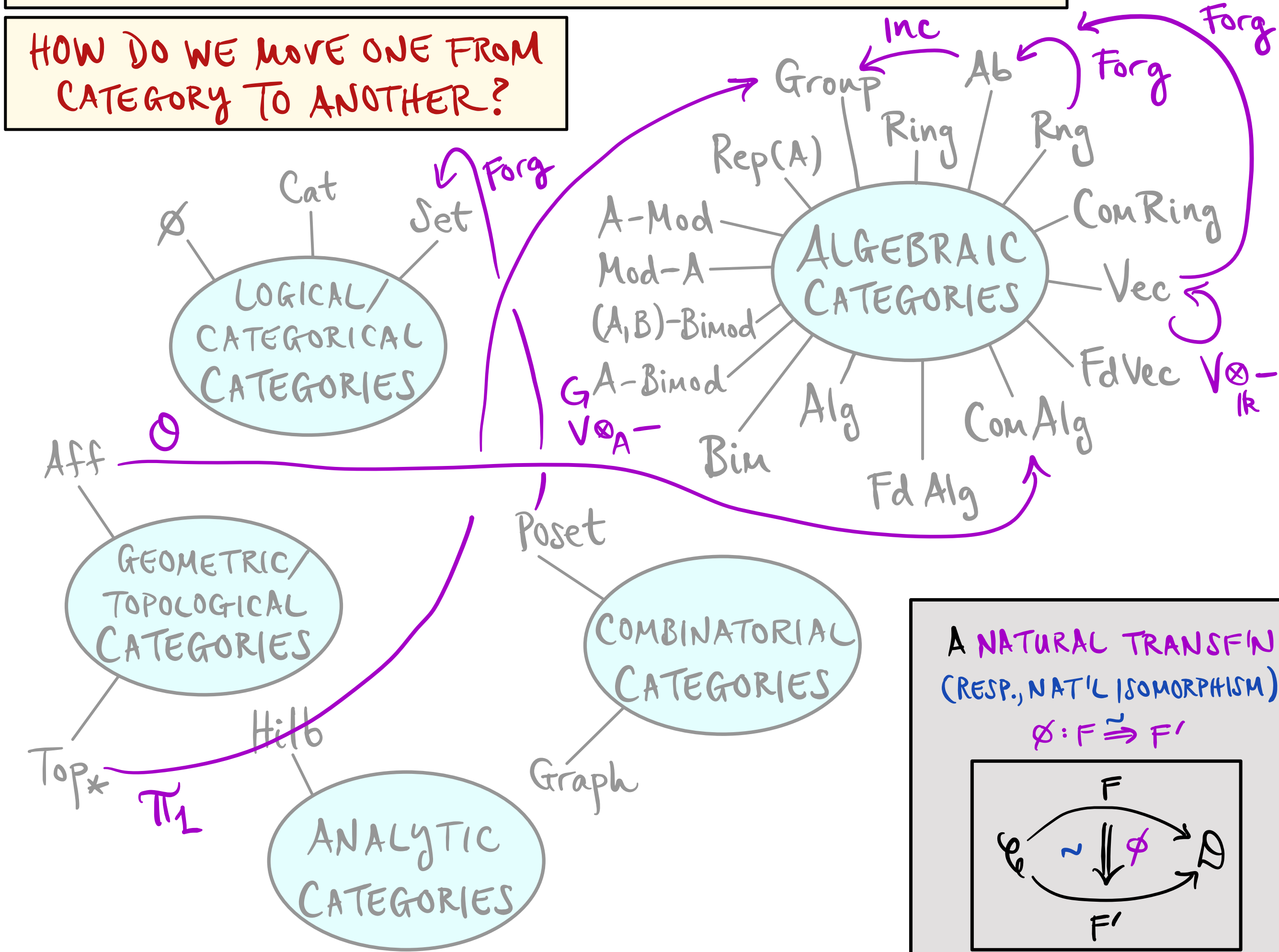
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) : F(X) \to F(Y) \in \mathcal{D}$
(RESP., $F(g) : F(Y) \to F(X) \in \mathcal{D}$)
$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

IDENTITY &

COMPOSED MORPHISMS




A **NATURAL TRANSF'N** (RESP., NAT'L ISOMORPHISM)
$\phi : F \xRightarrow{\sim} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

| | | | |
|---|---|---|---|
| $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$ | | | F **EMBEDDING**: F FAITH & INJ ON OBJS |
| F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$ | F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$ | F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$ | F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$ |

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

NOW SOME ANSWERS

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE SAID TO BE

**ISOMORPHIC**

IF $\exists$ FUNCTORS

$F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

SUCH THAT

$GF = Id_{\mathcal{C}}$

& $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

# I. ISOMORPHISM OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$
ARE SAID TO BE
ISOMORPHIC
IF $\exists$ FUNCTORS
$F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$
SUCH THAT
$GF = Id_{\mathscr{C}}$
& $FG = Id_{\mathscr{D}}$

WRITE $\mathscr{C} \cong \mathscr{D}$

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \text{Rep}(G)}$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE ISOMORPHIC IF $\exists$ FUNCTORS $F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}}$$

$$\& \ FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F : G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$\Big(\underset{\mathrm{Vec}}{\overset{}{V}}, \ \triangleright : G \times V \to V\Big) \longmapsto \Big(V, \ \rho_V : G \to GL(V)\Big)$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$F' : \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$\Big(\underset{\mathrm{Vec}}{V}, \ \rho : G \to GL(V)\Big) \longmapsto \Big(V, \ \triangleright_V : G \times V \to V\Big)$$

$$(g, v) \mapsto \rho_g(v)$$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE SAID TO BE

ISOMORPHIC

IF $\exists$ FUNCTORS

$F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

SUCH THAT

$GF = \mathrm{Id}_{\mathcal{C}}$

& $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$\underset{\underset{\mathrm{Vec}}{\cap}}{(V}, \underbrace{\triangleright: G \times V \to V}_{\text{= ACTION =}}) \longmapsto (V, \underbrace{\rho_V : G \to GL(V)}_{\text{= GROUP HOMOM. =}})$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$

$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$\underset{\underset{\mathrm{Vec}}{\cap}}{(V}, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V : G \times V \to V)$$

$$(g, v) \mapsto \rho_g(v)$$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE SAID TO BE

**ISOMORPHIC**

IF $\exists$ FUNCTORS

$F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

SUCH THAT

$GF = \mathrm{Id}_{\mathcal{C}}$

& $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$\Big(\underset{\mathrm{Vec}}{V}, \underbrace{\triangleright : G \times V \to V}_{\text{ACTION}}\Big) \longmapsto \Big(V, \underbrace{\rho_V : G \to GL(V)}_{\text{GROUP HOMOM.}}\Big)$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$\Big(\underset{\mathrm{Vec}}{V}, \rho : G \to GL(V)\Big) \longmapsto \Big(V, \triangleright_V : G \times V \to V\Big)$$

$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE **ISOMORPHIC** IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}}$$

$$\& \; FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

UPGRADE OF EXER. 1.13

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(\underset{\mathrm{Vec}}{\overset{}{V}}, \underbrace{\triangleright: G \times V \to V}_{\equiv \text{ ACTION } \equiv}) \longmapsto (V, \underbrace{\rho_V : G \to GL(V)}_{\equiv \text{ GROUP HOMOM.} \equiv})$$

$V \in \mathrm{Vec}$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F' : \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(V, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V : G \times V \to V)$$

$V \in \mathrm{Vec}$

$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

# I. ISOMORPHISM OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$ ARE SAID TO BE **ISOMORPHIC** IF $\exists$ FUNCTORS $F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$ SUCH THAT

$GF = \mathrm{Id}_{\mathscr{C}}$ & $FG = \mathrm{Id}_{\mathscr{D}}$

WRITE $\mathscr{C} \cong \mathscr{D}$

EXERCISE 2.30: SHOW –

$G\text{-Mod} \cong \mathrm{Rep}(G)$

$\cong \mathrm{Rep}(\Bbbk G)$

$\cong \Bbbk G\text{-Mod}$

UPGRADE OF EXER. 1.13

EXAMPLE: $G$ = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(V, \underbrace{\triangleright : G \times V \to V}_{\equiv\text{ ACTION }\equiv}) \mapsto (V, \underbrace{\rho_V : G \to GL(V)}_{\equiv\text{ GROUP HOMOM. }\equiv})$$

$V \in \mathrm{Vec}$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$
$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(V, \rho: G \to GL(V)) \mapsto (V, \triangleright_V : G \times V \to V)$$

$V \in \mathrm{Vec}$

$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE **ISOMORPHIC** IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}}$$
$$\& \; FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

UPGRADE OF EXER. 1.13

**EXAMPLE:** $G$ = GROUP

**GET:** $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \underbrace{\triangleright : G \times V \to V}_{\equiv\text{ ACTION }\equiv}) \longmapsto (V, \underbrace{\rho_V : G \to GL(V)}_{\equiv\text{ GROUP HOMOM. }\equiv})$$
$$g \mapsto [V \to V, \; v \mapsto g \triangleright v]$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$
$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \rho : G \to GL(V)) \longmapsto (V, \triangleright_V : G \times V \to V)$$
$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

EXERCISE 2.30: SHOW –

$G\text{-Mod} \cong \mathrm{Rep}(G)$

$\cong \mathrm{Rep}(\Bbbk G) \quad \ni (V, \underbrace{\rho : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)}_{\equiv\text{ ALG. MORPHISM }\equiv})$

$\cong \Bbbk G\text{-Mod} \quad \ni (V, \triangleright : \Bbbk G \times V \to V)$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ s.t. $GF = Id_{\mathcal{C}}$ & $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/ \Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

# I. ISOMORPHISM OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$ ARE **ISOMORPHIC** IF $\exists F : \mathscr{C} \to \mathscr{D}$ & $G : \mathscr{D} \to \mathscr{C}$ $\ni$ $GF = \mathrm{Id}_{\mathscr{C}}$ & $FG = \mathrm{Id}_{\mathscr{D}}$

WRITE $\mathscr{C} \cong \mathscr{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathscr{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathscr{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F : \mathrm{FdVec} \longrightarrow \mathscr{S}$ & $G : \mathscr{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

<u>BUT</u> $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V}$

## I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\therefore \mathrm{FdVec} \not\cong \mathcal{S}$

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER $\mathsf{FdVec}$ / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathsf{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS $\mathsf{FdVec}$ & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$$F : \mathsf{FdVec} \longrightarrow \mathcal{S} \qquad \& \qquad G : \mathcal{S} \longrightarrow \mathsf{FdVec}$$
$$V \longmapsto \Bbbk^{\oplus \dim_{\Bbbk} V} \qquad\qquad \Bbbk^{\oplus n} \longmapsto \Bbbk^{\oplus n}$$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \underbrace{\Bbbk^{\oplus \dim_{\Bbbk} V}}_{\cong V} \neq V$

$\therefore \mathsf{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t.

$GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad \Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t.

$GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$

= FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$

ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS

EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

**SKELETON OF $\mathcal{C}$** = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

**SKELETON OF $\mathcal{C}$** = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER $\mathsf{FdVec}$ $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathsf{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathsf{FdVec})$

PERHAPS $\mathsf{FdVec}$ & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathsf{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathsf{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathsf{FdVec} \not\cong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

# I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE ISOMORPHIC IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF=\mathrm{Id}_{\mathcal{C}}$ & $FG=\mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C}\cong\mathcal{D}$

SKELETON OF $\mathcal{C}$

= FULL SUBCATEGORY

$\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

<u>EXER. 2.33</u>

$\mathrm{Skel}(\mathcal{C})\cong\mathcal{C} \iff \mathrm{Skel}(\mathcal{C})=\mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n\in\mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS

EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n\in\mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad \Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

<u>BUT</u> $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = Id_{\mathcal{C}}$ & $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $Skel(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $Obj(\mathcal{C})$

EXER. 2.33

$Skel(\mathcal{C}) \cong \mathcal{C} \iff Skel(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec /$\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $FdVec_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = Skel(FdVec)$

$F: FdVec \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow FdVec$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore FdVec \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

SKELETON OF $\mathcal{C}$ $\equiv$ FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \simeq \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \neq \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \wr"] & \mathcal{D} \end{tikzcd}$$

$G \equiv$ "QUASI-INVERSE" OF $F$

SKELETON OF $\mathcal{C}$

$\equiv$ FULL SUBCATEGORY $Skel(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $Obj(\mathcal{C})$

EXER. 2.33

$Skel(\mathcal{C}) \cong \mathcal{C} \iff Skel(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\text{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = Skel(\text{FdVec})$

$F : \text{FdVec} \longrightarrow \mathcal{S}$ & $G : \mathcal{S} \longrightarrow \text{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ $\text{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$

$\equiv$ FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/ \Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F : \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G : \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \simeq \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \not\simeq \mathcal{S}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ $\equiv$ FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore \mathrm{FdVec} \simeq \mathcal{S}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER $\mathrm{FdVec}$ $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F : \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G : \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore \mathrm{FdVec} \simeq \mathcal{S}$

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{D} \end{tikzcd}$$

**SKELETON OF $\mathcal{C}$** = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

**EXER. 2.33**

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

$\mathrm{Skel}(\mathcal{C}) \simeq \mathcal{C}$ ALWAYS

$\mathcal{C} \simeq \mathcal{D} \iff \mathrm{Skel}(\mathcal{C}) \cong \mathrm{Skel}(\mathcal{D})$

CONSIDER $\mathsf{FdVec}$ $/ \Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathsf{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathsf{FdVec})$

$F : \mathsf{FdVec} \longrightarrow \mathcal{S}$ & $G : \mathcal{S} \longrightarrow \mathsf{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}.$

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore \mathsf{FdVec} \simeq \mathcal{S}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

LIKE TWO STRUCTURES ARE THE SAME

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

$$\underbrace{\hspace{6em}}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

## II. EQUIVALENCE OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$ ARE **EQUIVALENT** IF $\exists\, F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$ s.t. $GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

WRITE $\mathscr{C} \simeq \mathscr{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

$\Updownarrow$

$\exists$ A BIJECTIVE STRUCTURE MAP FROM ONE TO THE OTHER

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni.$

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

$\Updownarrow$

$\exists$ A BIJECTIVE STRUCTURE MAP FROM ONE TO THE OTHER

A CHARACTERIZATION IN TERMS OF...

$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\longmapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**: $\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C} \ni.\ Y\cong F(X)$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \longmapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g \longmapsto F(g)$$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:

$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$$
$$g \mapsto F(g)$$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

**THEOREM**
$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni.$

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g\longmapsto F(g)$$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y\in\mathcal{D}$, $\exists X\in\mathcal{C}$ $\ni.$ $Y\cong F(X)$

**THEOREM**

$\mathcal{C}\simeq\mathcal{D}$ $\iff$ $\exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/($\Rightarrow$) SAY $\exists\,\phi:\mathrm{Id}_{\mathcal{C}}\overset{\sim}{\Rightarrow} GF$ & $\psi: FG\overset{\sim}{\Rightarrow}\mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: $F$ IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\xrightarrow[\sim]{\psi_Y} Y$ $\Rightarrow$ $F$ ESS. SURJ.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

**THEOREM**

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi:\mathrm{Id}_{\mathcal{C}}\overset{\sim}{\Rightarrow} GF$ & $\psi: FG\overset{\sim}{\Rightarrow}\mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g:X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THIS IMPLIES F IS FAITHFUL

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ $(\Rightarrow)$ SAY $\exists\, \phi: \mathrm{Id}_{\mathcal{C}} \xRightarrow{\sim} GF$ & $\psi: FG \xRightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

**CLAIM:** $F$ IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall\, g: X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r] & GF(X') \arrow[u, "\cong", "\phi_{X'}^{-1}"'] \end{tikzcd}$$

$GF(g) \overset{\text{IF}}{=} GF(\tilde{g})$

THIS IMPLIES $F$ IS FAITHFUL

$[F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g})]$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋. $GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

---

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)\to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y\in\mathcal{D}$, $\exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

---

**THEOREM** $\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi: \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: $F$ IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g: X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g \overset{\text{THEN}}{=} \tilde{g}"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g) \overset{\text{IF}}{=} GF(\tilde{g})"'] & GF(X') \arrow[u, "\cong", "\phi_{X'}^{-1}"'] \end{tikzcd}$$

THIS IMPLIES $F$ IS FAITHFUL

$$\left[ \begin{array}{l} F(g)=F(\tilde{g}) \Rightarrow GF(g)=GF(\tilde{g}) \\ \qquad\qquad\quad \Rightarrow g=\tilde{g} \end{array} \right]$$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}:Hom_{\mathcal{C}}(X,Y)\longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y\in\mathcal{D}$, $\exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

**THEOREM** $\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG\overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: $F$ IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall\, g:X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g=\tilde{g}$ IF $GF(g) = GF(\tilde{g})$

THIS IMPLIES $F$ IS FAITHFUL

$$\left[\begin{array}{l} F(g)=F(\tilde{g}) \Rightarrow GF(g)=GF(\tilde{g}) \\ \qquad\qquad\quad \Rightarrow g=\tilde{g} \end{array}\right]$$

SWAPPING $\phi$ WITH $\psi \Rightarrow G$ IS FAITHFUL.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∍.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∍. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow$ F ESS. SURJ.

NOW $\forall g: X \to X' \in \mathcal{C}$, GET

THEN $g = \tilde{g}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g) \,=\, GF(\tilde{g}) \text{ IF}"'] & GF(X') \arrow[u, "\cong"', "\phi_{X'}^{-1}"] \end{tikzcd}$$

THIS IMPLIES F IS FAITHFUL

$$\left[ F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \Rightarrow g = \tilde{g} \right]$$

SWAPPING $\phi$ WITH $\psi \Rightarrow$ G IS FAITHFUL.

TAKE $h: F(X) \to F(X') \in \mathcal{D}$.

BUILD $g: X \xrightarrow{\phi_X} GF(X) \xrightarrow{G(h)} GF(X') \xrightarrow{\phi_{X'}^{-1}} X' \in \mathcal{C}$.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

---

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

---

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: $F$ IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g: X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] & X' \\ GF(X) \arrow[r, "GF(g)"'] \arrow[ur, phantom, "2"] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$

$GF(g) \;\|\;$ IF $GF(\tilde{g})$

THIS IMPLIES $F$ IS FAITHFUL

$\left[ \begin{array}{l} F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \\ \qquad\qquad \Rightarrow g = \tilde{g} \end{array} \right]$

SWAPPING $\phi$ WITH $\psi \Rightarrow G$ IS FAITHFUL.

TAKE $h: F(X) \to F(X') \in \mathcal{D}$.

BUILD $g: X \xrightarrow{\phi_X} GF(X) \xrightarrow{G(h)} GF(X') \xrightarrow{\phi_{X'}^{-1}} X' \in \mathcal{C}$.

THEN $GF(g) = G(h)$. $G$ FAITHFUL $\Rightarrow F(g) = h$

& $F$ IS FULL.

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∍.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∍. $Y \cong F(X)$

**THEOREM** $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C}\to\mathcal{D}$

PF/($\Rightarrow$) SAY $\exists\, \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow$ F ESS. SURJ.

NOW $\forall\, g: X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$

$\| $ IF $GF(\tilde{g})$

THIS IMPLIES F IS FAITHFUL

$$\left[\begin{aligned} F(g) = F(\tilde{g}) &\Rightarrow GF(g) = GF(\tilde{g}) \\ &\Rightarrow g = \tilde{g} \end{aligned}\right]$$

SWAPPING $\phi$ WITH $\psi \Rightarrow$ G IS FAITHFUL.

TAKE $h: F(X) \to F(X') \in \mathcal{D}$.

BUILD $g: X \xrightarrow{\phi_X} GF(X) \xrightarrow{G(h)} GF(X') \xrightarrow{\phi_{X'}^{-1}} X' \in \mathcal{C}$.

THEN $GF(g) = G(h)$. G FAITHFUL $\Rightarrow F(g) = h$

& F IS FULL.

## II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g \longmapsto F(g)$$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F : \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$
s.t.
$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$\qquad g\longmapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\qquad \forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

**THEOREM**
$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$$F \text{ ESS. SURJ} \Rightarrow \forall Y\in\mathcal{D}\ \ \exists \underset{\underset{G(Y)}{\|}}{z_Y}\in\mathcal{C} \text{ s.t. } F(z_Y)\underset{\underset{=:\psi_Y}{\uparrow}}{\cong} Y$$

LABELLING: $G(Y)$, $=:\psi_Y$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

PF/ ($\Longleftarrow$) TAKE $F: \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow$ $\forall Y \in \mathcal{D}$ $\exists \underset{\substack{\| \\ G(Y)}}{Z_Y} \in \mathcal{C}$ s.t. $F(Z_Y) \underset{=:\psi_Y}{\cong} Y$

F FULLY FAITHFUL $\Rightarrow$ $\forall\ g: Y \to Y' \in \mathcal{D}$

$\exists!$ MORPHISM $G(Y) \underset{\substack{\| \\ G(g)}}{\longrightarrow} G(Y') \in \mathcal{C}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

---

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$\qquad g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\qquad \forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

---

**THEOREM**

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Longleftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$F$ ESS. SURJ $\Rightarrow \forall Y\in\mathcal{D}\ \ \exists Z_Y\in\mathcal{C}$ s.t. $F(Z_Y)\cong Y$

$Z_Y = G(Y)$, $\quad \cong\ =: \psi_Y$

$F$ FULLY FAITHFUL $\Rightarrow \forall\, g: Y\to Y' \in\mathcal{D}$

$\exists!$ MORPHISM $G(Y)\to G(Y')\in\mathcal{C}$, $\quad =: G(g)$

NEED TO SHOW

- ASSIGNMENTS $Y\mapsto G(Y)$, $g\mapsto G(g)$ MAKE A FUNCTOR $G:\mathcal{D}\to\mathcal{C}$

- $\exists$ NATURAL ISOM $\psi: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

- $\exists$ NATURAL ISOM $\phi: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

---

$F_{X,Y} : Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$\qquad g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

---

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F : \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$F$ ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D} \ \exists \underset{\substack{\| \\ G(Y)}}{Z_Y} \in \mathcal{C} \ni. F(Z_Y) \underset{=: \, \psi_Y}{\cong} Y$

$F$ FULLY FAITHFUL $\Rightarrow \forall g : Y \to Y' \in \mathcal{D}$
$\exists!$ MORPHISM $\underset{\substack{\| \\ G(g)}}{G(Y) \to G(Y')} \in \mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $Y \mapsto G(Y)$, $g \mapsto G(g)$ MAKE A FUNCTOR $G : \mathcal{D} \to \mathcal{C}$
  - $F$ FAITHFUL
- $\exists$ NATURAL ISOM $\psi : FG \Rightarrow Id_{\mathcal{D}}$
- $\exists$ NATURAL ISOM $\phi : Id_{\mathcal{C}} \Rightarrow GF$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F : \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$F$ ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D}\ \ \exists \underbrace{Z_Y}_{=: G(Y)} \in \mathcal{C}$ s.t. $F(Z_Y) \underbrace{\cong}_{=: \psi_Y} Y$

$F$ FULLY FAITHFUL $\Rightarrow \forall\ g : Y \to Y' \in \mathcal{D}$
$\exists!$ MORPHISM $\underbrace{G(Y) \to G(Y')}_{=: G(g)} \in \mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $Y \mapsto G(Y)$, $g \mapsto G(g)$ MAKE A FUNCTOR $G : \mathcal{D} \to \mathcal{C}$
  ↰ F FAITHFUL
- $\exists$ NATURAL ISOM $\psi : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ WITH COMPONENTS $\psi_Y$
- $\exists$ NATURAL ISOM $\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

---

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \longmapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

---

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F : \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$F$ ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D} \ \exists \underset{G(Y)}{\underset{\shortparallel}{z_Y}} \in \mathcal{C} \ \ni. \ F(z_Y) \underset{=:\psi_Y}{\underset{\uparrow}{\cong}} Y$

$F$ FULLY FAITHFUL $\Rightarrow \forall g : Y \to Y' \in \mathcal{D}$
$\exists!$ MORPHISM $G(Y) \underset{G(g)}{\underset{\shortparallel}{\to}} G(Y') \in \mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $Y \mapsto G(Y)$, $g \mapsto G(g)$ MAKE A FUNCTOR $G : \mathcal{D} \to \mathcal{C}$
  ↰ F FAITHFUL
- $\exists$ NATURAL ISOM $\psi : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ WITH COMPONENTS $\psi_Y$
- $\exists$ NATURAL ISOM $\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$
  USING COMPONENTS $\psi^{-1}_{F(X)} : F(X) \to FGF(X)$ & F FULLY FAITHFUL

# II. EQUIVALENCE OF CATEGORIES

DETAILS ≡ EXER 2.34

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋.

$GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y)\longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g\longmapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

**THEOREM** $\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y\in\mathcal{D}\ \exists \underbrace{Z_Y}_{=\,G(Y)}\in\mathcal{C}$ ∋. $F(Z_Y)\underbrace{\cong}_{=:\,\psi_Y} Y$

F FULLY FAITHFUL $\Rightarrow \forall\, g: Y\to Y'\in\mathcal{D}$
$\exists!$ MORPHISM $\underbrace{G(Y)\to G(Y')}_{=\,G(g)}\in\mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $Y\mapsto G(Y)$, $g\mapsto G(g)$ MAKE A FUNCTOR $G:\mathcal{D}\to\mathcal{C}$ ← F FAITHFUL
- $\exists$ NATURAL ISOM $\psi: FG\Rightarrow Id_{\mathcal{D}}$ WITH COMPONENTS $\psi_Y$
- $\exists$ NATURAL ISOM $\phi: Id_{\mathcal{C}}\Rightarrow GF$ USING COMPONENTS $\psi^{-1}_{F(X)}: F(X)\to FGF(X)$ & F FULLY FAITHFUL //

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

$F$ FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ ESS. SURJ:

$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

## TOWARD EXAMPLES –

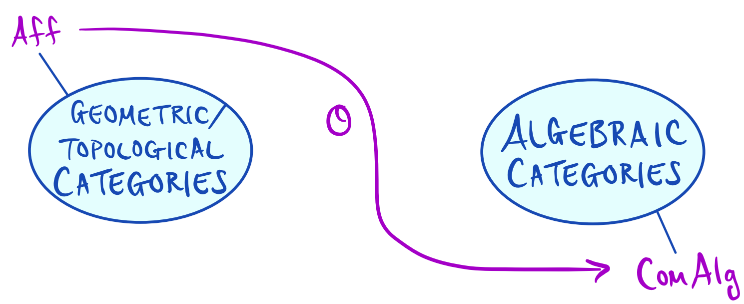
SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO $0$

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

$X$: AFFINE VARIETY

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF $X$



# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

$F$ FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

## TOWARD EXAMPLES –

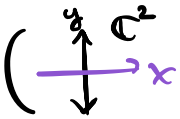
SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO $0$

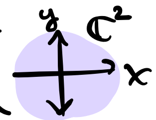
$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF $X$

Ex. $n=2$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

$\mathcal{O}\Big($  $\Big) = \mathbb{C}[x,y]$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

## TOWARD EXAMPLES –

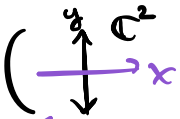
$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

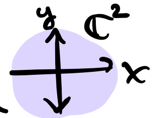
AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

COORDINATE ALGEBRA OF $X$

Ex. $n=2$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

$\mathcal{O}\Big($  $\Big) = \mathbb{C}[x,y]$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

**THEOREM**
$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

## TOWARD EXAMPLES—

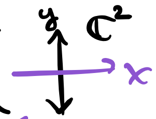
SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\left(\substack{\text{IDEAL OF}\\ \text{POLY'LS}\\ \text{DEFINING } X}\right)}$

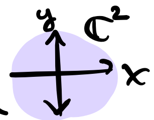
AFFINE VARIETY — COORDINATE ALGEBRA OF $X$

Ex. $n=2$

$\mathcal{O}\Big($   $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

PROJECTION

$\mathcal{O}\Big($   $\Big) = \mathbb{C}[x,y]$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

= CONTRAVARIANT =

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

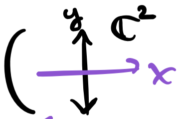
$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY

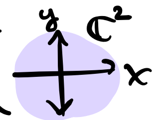
COORDINATE ALGEBRA OF $X$

Ex. $n=2$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

PROJECTION

$\mathcal{O}\Big($  $\Big) = \mathbb{C}[x,y]$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∍.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \longmapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ ∍. $Y\cong F(X)$

**THEOREM**

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

≡ CONTRAVARIANT ≡

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

$X$: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF $X$

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\quad \dots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C} \ni.\ Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

AFFINE VARIETY

COORDINATE ALGEBRA OF $X$

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\quad \dots\ \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**: $\forall Y\in\mathcal{D}, \exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

**THEOREM**

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

≡ CONTRAVARIANT ≡

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

$X$: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO $0$

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF $X$

$$\mathrm{Aff}_{\mathbb{C}} \simeq \left(\mathrm{Fg} \dots \mathrm{ComAlg}_{\mathbb{C}}\right)^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C}
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D}
\end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: Aff_{\mathbb{C}} \longrightarrow ComAlg_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY

COORDINATE ALGEBRA OF X

$$Aff_{\mathbb{C}} \simeq (Fg \dots ComAlg_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x=y=0$ (ORIGIN)

$\parallel$

THE SHAPE $x^2=y=0$ (ORIGIN)

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

**THEOREM** $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n] \Big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\mathrm{Fg} \dots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x = y = 0$ (ORIGIN) BUT $\mathbb{C}[x,y]/(x,y)$

$\|$ $\qquad\qquad\qquad\qquad \not\cong$

THE SHAPE $x^2 = y = 0$ (ORIGIN) $\quad \mathbb{C}[x,y]/(x^2,y)$ AS ALGS

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

$F$ FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ ESS. SURJ: $\forall Y\in\mathcal{D}$, $\exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

**THEOREM**

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

CONTRAVARIANT

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \Big/ \left(\begin{smallmatrix}\text{IDEAL OF} \\ \text{POLY'LS} \\ \text{DEFINING } X\end{smallmatrix}\right)$

AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

COORDINATE ALGEBRA OF $X$

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\mathrm{Fg}\underbrace{\mathrm{Red}}_{\text{"REDUCED"}}\mathrm{ComAlg}_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x=y=0$ (ORIGIN)

$\parallel$

THE SHAPE $x^2=y=0$ (ORIGIN)

BUT $\mathbb{C}[x,y]/(x,y) \ncong \mathbb{C}[x,y]/(x^2,y)$ AS ALGS

MAKES THIS CHOICE TO CORRESP. TO ORIGIN

NO NILPOTENT ELEMENTS

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \Big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

AFFINE VARIETY

COORDINATE ALGEBRA OF X

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\mathrm{FgRedComAlg}_{\mathbb{C}})^{op}$$

ALSO HAVE

SOMETHING ELSE GEOMETRIC $_{\mathbb{C}} \simeq (\mathrm{ComAlg}_{\mathbb{C}})^{op}$

# II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

$F$ **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

$F$ **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

$F$ **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

$F$ **ESS. SURJ**:
$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

**THEOREM**

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n] \big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

AFFINE VARIETY — COORDINATE ALGEBRA OF $X$

$$\mathrm{Aff}_{\mathbb{C}} \simeq \left(\mathrm{FgRedComAlg}_{\mathbb{C}}\right)^{op}$$

ALSO HAVE

BEYOND SCOPE OF LECTURE

$$\mathrm{Scheme}_{\mathbb{C}} \simeq \left(\mathrm{ComAlg}_{\mathbb{C}}\right)^{op}$$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni$

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

## NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

## NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

## NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

# NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

WEAKEN

$A \underset{\text{MORITA}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$

IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{\text{reg}}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{\text{reg}}$ AS $B$-BIMODULES.

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff$ $\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longleftarrow$)

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Leftarrow$)

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

NOW $\forall\, M \in A\text{-Mod}$, GET:

$$GF(M) = G(Q \otimes_A M) = P \otimes_B (Q \otimes_A M)$$

$$\cong (P \otimes_B Q) \otimes_A M$$

$\boxed{\text{MODIFY EXER. 1.186}}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

∋.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod}\underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ ∋.

$P\otimes_B Q\cong A_{reg}$ AS $A$-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longleftarrow$)

TAKE $F:=Q\otimes_A - : A\text{-Mod}\longrightarrow B\text{-Mod}$

$G:=P\otimes_B - : B\text{-Mod}\longrightarrow A\text{-Mod}$

NOW $\forall\, M\in A\text{-Mod}$, GET:

$$GF(M)=G(Q\otimes_A M)=P\otimes_B(Q\otimes_A M)$$

$$\cong (P\otimes_B Q)\otimes_A M \underset{\text{HYPOTHESIS}}{\cong} A_{reg}\otimes_A M \underset{\boxed{\text{EXER. 1.18a}}}{\cong} M.$$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$

$$GF \cong \mathrm{Id}_{\mathcal{C}} \quad \& \quad FG \cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Leftarrow$)

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

NOW $\forall M \in A\text{-Mod}$, GET:

$$GF(M) = G(Q \otimes_A M) = P \otimes_B (Q \otimes_A M)$$

$$\cong (P \otimes_B Q) \otimes_A M \cong A_{reg} \otimes_A M \cong M.$$

$\therefore GF \cong \mathrm{Id}_{A\text{-Mod}}$.

LIKEWISE, $FG \cong \mathrm{Id}_{B\text{-Mod}}$. //

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ $(\Longrightarrow)$ GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET $A^{op} \cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg})$

$\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg}))$

$\cong \mathrm{End}_{B\text{-Mod}}({}_B Q)$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET $A^{op} \cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg})$

$\boxed{\text{EXER. 1.26}}$

$\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg}))$

F FULLY FAITHFUL

$\cong \mathrm{End}_{B\text{-Mod}}({}_B Q)$

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ ∋.

$$GF \cong \mathrm{Id}_{\mathcal{C}} \quad \& \quad FG \cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ ∋.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET
$$\begin{aligned} A^{op} &\cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg}) \\ &\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg})) \\ &\cong \mathrm{End}_{B\text{-Mod}}({}_B Q) \end{aligned}$$

$f : A^{op} \xrightarrow{\ \sim\ } \mathrm{End}_{B\text{-Mod}}({}_B Q)$

DEFINE $q \triangleleft a := f(a)(q)$

$\rightsquigarrow$ ${}_B Q \in (B, A)\text{-Bimod}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$

IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ $(\Longrightarrow)$ GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B, A)\text{-Bimod}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$
ARE **EQUIVALENT** IF
$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$
s.t.
$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$
IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$
$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.
$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES
& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B, A)\text{-Bimod}$

CLAIM: $F \cong Q \otimes_A -$ AS FUNCTORS

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ ∋.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B, A)\text{-Bimod}$

CLAIM : $F \cong Q \otimes_A -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO:

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A, X) \xrightarrow[\cong]{F} \mathrm{Hom}_{B\text{-Mod}}(F(A), F(X)) = \mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

F FULLY FAITHFUL

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

CLAIM : $F \cong Q \otimes_A -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO :

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A, X) \xrightarrow{F} \mathrm{Hom}_{B\text{-Mod}}(F(A), F(X))$$
$$\| $$
$$\mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

TENSOR-HOM ADJUNCTION

$\mathrm{Hom}_{B\text{-Mod}}(Q \otimes_A X, Y) \cong \mathrm{Hom}_{A\text{-Mod}}(X, \mathrm{Hom}_{B\text{-Mod}}(Q, Y))$

GET ISO : $\sigma_X' : Q \otimes_A X \longrightarrow F(X)$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ $(\Longrightarrow)$ GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

CLAIM: $F \cong Q \otimes_B -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO:

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A,X) \xrightarrow{F} \mathrm{Hom}_{B\text{-Mod}}(F(A),F(X))$$
$$\| $$
$$\mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

TENSOR-HOM ADJUNCTION

$\mathrm{Hom}_{B\text{-Mod}}(Q \otimes_A X, Y) \cong \mathrm{Hom}_{A\text{-Mod}}(X, \mathrm{Hom}_{B\text{-Mod}}(Q,Y))$

GET ISO: $\sigma'_X : Q \otimes_A X \longrightarrow F(X) \rightsquigarrow Q \otimes_A - \underset{\sim}{\overset{\sigma}{\Rightarrow}} F$

## III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$

IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

# III. Morita Equivalence

$\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if $\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$.

$$GF \cong \mathrm{Id}_{\mathcal{C}} \quad \& \quad FG \cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"{name=U}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'{name=D}] & \mathcal{C} \arrow[phantom, from=U, to=D, "\Downarrow\sim"] \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"{name=U}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'{name=D}] & \mathcal{D} \arrow[phantom, from=U, to=D, "\Downarrow\sim"] \end{tikzcd}$$

Write $\mathcal{C} \simeq \mathcal{D}$

Take $\Bbbk$-algs $A$ & $B$

$A$ is **Morita equiv.** to $B$ if $A\text{-Mod} \underset{\text{linear}}{\simeq} B\text{-Mod}$.

That is, $A$ & $B$ have the same rep thy

**Morita's Theorem** $A$ is Morita equivalent to $B$

$\iff \exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ as $A$-bimodules

& $Q \otimes_A P \cong B_{reg}$ as $B$-bimodules.

Pf/ ($\Rightarrow$) Given equivalence $F : A\text{-Mod} \to B\text{-Mod}$

Have $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

Have $F \cong Q \otimes_A -$ as functors

Now $\exists\, G : B\text{-Mod} \to A\text{-Mod}$ with

$\phi : \mathrm{Id}_{A\text{-Mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B, A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

NOW $\exists\, G : B\text{-Mod} \to A\text{-Mod}$ WITH

$\phi : \mathrm{Id}_{A\text{-Mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

HAVE $P := G({}_B B_{reg}) \in (A, B)\text{-Bimod}$

HAVE $G \cong P \otimes_B -$ AS FUNCTORS

# III. MORITA EQUIVALENCE

$\mathscr{C}$ AND $\mathscr{D}$ ARE **EQUIVALENT** IF $\exists\, F : \mathscr{C} \to \mathscr{D}$ & $G : \mathscr{D} \to \mathscr{C}$ $\ni$.

$$GF \cong \mathrm{Id}_{\mathscr{C}} \quad \& \quad FG \cong \mathrm{Id}_{\mathscr{D}}$$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B, A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

NOW $\exists\, G : B\text{-Mod} \to A\text{-Mod}$ WITH

$\phi : \mathrm{Id}_{A\text{-Mod}} \xRightarrow{\sim} GF$ & $\psi : FG \xRightarrow{\sim} \mathrm{Id}_{B\text{-Mod}}$

HAVE $P := G({}_B B_{reg}) \in (A, B)\text{-Bimod}$

HAVE $G \cong P \otimes_B -$ AS FUNCTORS

GET $\phi_A : A \xrightarrow{\sim} GF(A) \cong P \otimes_B Q$ AS $A$-BIMODS

& $\psi_B^{-1} : B \xrightarrow{\sim} FG(B) \cong Q \otimes_A P$ AS $B$-BIMODS $//\!/$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS **MORITA EQUIV.** TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

DETAILS = EXER. 2.35

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

NOW $\exists\, G : B\text{-Mod} \to A\text{-Mod}$ WITH

$\phi : \mathrm{Id}_{A\text{-Mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

HAVE $P := G({}_B B_{reg}) \in (A,B)\text{-Bimod}$

HAVE $G \cong P \otimes_B -$ AS FUNCTORS

GET $\phi_A : A \xrightarrow{\sim} GF(A) \cong P \otimes_B Q$ AS $A$-BIMODS

& $\psi_B^{-1} : B \xrightarrow{\sim} FG(B) \cong Q \otimes_A P$ AS $B$-BIMODS //

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

**MORITA'S THEOREM** A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

**MAIN EXAMPLE**

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

## MAIN EXAMPLE

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

VIA

$P = \{ (a_1, \dots, a_n) \mid a_i \in A \} \in (A, \mathrm{Mat}_n(A))\text{-Bimod}$

$Q = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \,\middle|\, a_i \in A \right\} \in (\mathrm{Mat}_n(A), A)\text{-Bimod}$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

## MAIN EXAMPLE

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

VIA

$P = \{(a_1, \dots, a_n) \mid a_i \in A\} \in (A, \mathrm{Mat}_n(A))\text{-Bimod}$

$a' \triangleright (a_1, \dots, a_n) := (a'a_1, \dots, a'a_n)$

$\underbrace{(a_1, \dots, a_n)}_{\underline{a}} \triangleleft A' := \underline{a}\, A'$ MATRIX MULTIP'N

$Q = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle|\, a_i \in A \right\} \in (\mathrm{Mat}_n(A), A)\text{-Bimod}$

LIKEWISE

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

**MAIN EXAMPLE**

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni$.

$GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P\otimes_B Q\cong A_{reg}$ AS $A$-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS $B$-BIMODULES.

**MAIN EXAMPLE**

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

**EXER. 2.38** $A\sim_{MOR} B \implies Z(A)\cong Z(B)$

CENTER OF A

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

## MAIN EXAMPLE

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{\text{MOR}} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

**EXER. 2.38** $A \sim_{\text{MOR}} B \implies Z(A) \cong Z(B)$

CENTER OF A

# III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> $A$ IS MORITA EQUIVALENT TO $B$

$\iff$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

<u>MAIN EXAMPLE</u>

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{\text{MOR}} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

<u>EXER. 2.38</u> $A \sim_{\text{MOR}} B \;\underset{\not\Leftarrow}{\Longrightarrow}\; Z(A) \cong Z(B)$

IN GENERAL CENTER OF A

# III. MORITA EQUIVALENCE

$\mathscr{C}$ AND $\mathscr{D}$

ARE EQUIVALENT IF

$\exists\, F : \mathscr{C} \to \mathscr{D}$ & $G : \mathscr{D} \to \mathscr{C}$

$\ni$

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

TAKE $\Bbbk$-ALGS $A$ & $B$

$A$ IS MORITA EQUIV. TO $B$ IF $A\text{-Mod} \underset{\text{LINEAR}}{\simeq} B\text{-Mod}$.

THAT IS, $A$ & $B$ HAVE THE SAME REP THY

**MORITA'S THEOREM** $A$ IS MORITA EQUIVALENT TO $B$

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS $A$-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS $B$-BIMODULES.

## MAIN EXAMPLE

$A$ IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{\text{MOR}} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

EXER. 2.38 $A \sim_{\text{MOR}} B$ $\underset{\not\Longleftarrow}{\Longrightarrow}$ $Z(A) \cong Z(B)$

IN GENERAL

CENTER OF A

FOR COMM. ALGS $C, C'$: $C \sim_{\text{MOR}} C' \Longleftrightarrow Z(C) \cong Z(C')$

## ≡ SUMMARY ≡

**NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS**

$A = B$
EQUALITY OF ALGEBRAS

$\xrightarrow{\text{WEAKEN}}$

$A \cong B$
ISOMORPHISM OF ALGEBRAS

$\xrightarrow{\text{WEAKEN}}$

$A \underset{\text{MORITA}}{\sim} B$
"EQUIVALENCE" OF ALGEBRAS

## ≡ SUMMARY ≡

### NOTION OF SAMENESS FOR CATEGORIES

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

### NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

WEAKEN

$A \underset{\text{MORITA}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

## ≡ SUMMARY ≡

### NOTION OF SAMENESS FOR CATEGORIES

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

WEAKEN

NEXT TIME: ADJUNCTION

### NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

WEAKEN

$A \underset{\text{MORITA}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

# LECTURE #9

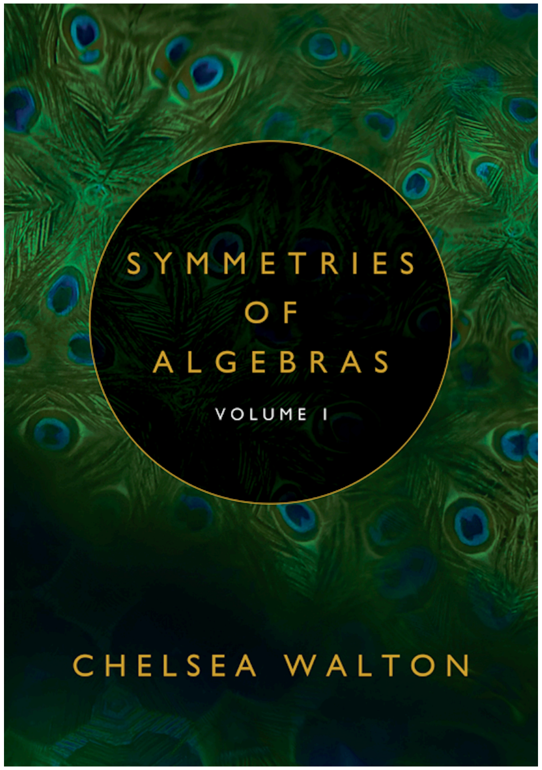
TOPICS:

- ~~I.~~ ISOMORPHISM OF CATEGORIES (§2.4.1)
- ~~II.~~ EQUIVALENCE OF CATEGORIES (§§2.4.2–2.4.3)
- ~~III.~~ MORITA EQUIVALENCE (§2.4.3)

NEXT TIME: ADJUNCTION

# Enjoy this lecture?
# You'll enjoy the textbook!

## C. Walton's "Symmetries of Algebras, Volume 1" (2024)




**Available for purchase at :**

**619 Wreath (at a discount)**

**https://www.619wreath.com/**

**Also on Amazon**
**&**
**Google Play**

Lecture #9 keywords: equivalence of categories, isomorphism of categories, Morita equivalence of algebras, Morita's Theorem, skeleton