

UNIVERSAL QUANTUM SEMIGROUPS ArXiv: 2008.00606

Joint with Hongdi Huang, Elizabeth Wicks, Robert Won.

$/k$ field.

- IN ALGEBRAIC QUANTUM SYMMETRY -



WE'LL PLAY GAME I for :

All "actions" here will preserve grading of A

$A = N$ -graded, locally finite k -algebra
 $(A = A_0 \oplus A_1 \oplus A_2 \oplus \dots ; \dim_k A_i < \infty)$
 ↑
 more on A_0 later

A as above

CLASSICAL
SYMMETRY

commutative
 $\neq$
 connected ($A_0 = k$)

Ex. $[k[x_1, \dots, x_n]]$

⋮

Hopf-type gadget H

group (acting by automorphisms)
 ... or more generally ...
 semigroup (acting by endomorphisms)

Ex. kG , $G \leq GL_n(k)$ finite that acts
 -or-
 kG , $G \leq \text{Mat}_n(k)$ finite that acts
 Dually...
 $\mathcal{O}(G)$ that coacts

QUANTUM SYMMETRY

"non"
= not necessarily

deform
↓
* noncommutative
&
* connected

Ex. $k_q[x_1, \dots, x_n]$ for $q \in k^\times$

↓ 'expand the base'
✓

WEAK QUANTUM SYMMETRY

* noncommutative
* &
* nonconnected
($\dim_k A_0$ could be > 1)

Ex. $A = kQ$, path algebra for a finite quiver $Q = (Q_0, Q_1, s, t: Q_1 \rightarrow Q_0)$

finite vertex set
finite arrow set
source, target maps

Here, $A_0 = kQ_0$

deform

quantum group $\equiv$ Hopf algebra that coacts
... or more generally ..

semi quantum group $\equiv$ bi algebra that coacts

Ex. $Q_q(G)$ that coacts

↓ 'expand the base'
✓

quantum group^{oid weak} $\equiv$ Hopf algebra that coacts
... or more generally ..

semi quantum group^{oid weak} $\equiv$ bi algebra that coacts

Ex. $\mathcal{H}(Q)$, weak bialgebra $\equiv$
Hayashi's face algebra
attached to Q
(defined later)

WEAK BI/HOPF ALGEBRAS : crash course

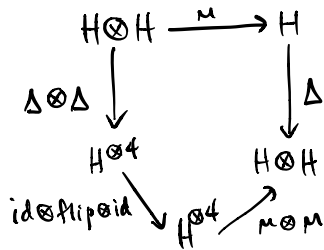
A weak bialgebra over k is a tuple:

$$(H, m: H \otimes H \rightarrow H, u: k \rightarrow H, \Delta: H \rightarrow H \otimes H, \varepsilon: H \rightarrow k)$$

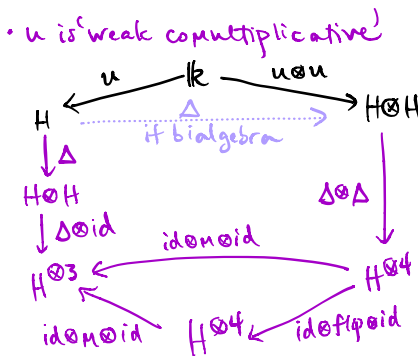
$\uparrow$ k -vs $\nwarrow \uparrow \nearrow$ k -linear maps

so that

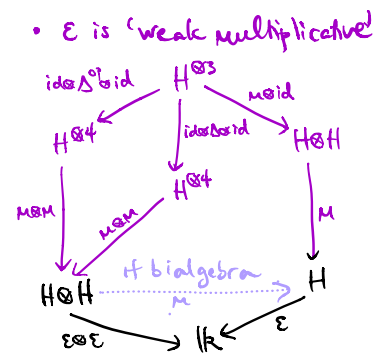
- (H, m, u) is an associative & unital k -algebra
- (H, Δ, ε) is a coassociative & counital k -coalgebra
- Δ is multiplicative
- u is 'weak comultiplicative'
- ε is 'weak multiplicative'



Commutative



Commutative



Commutative

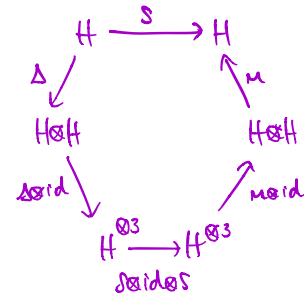
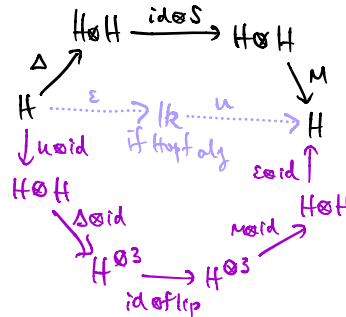
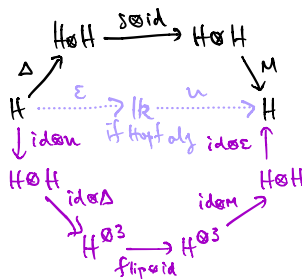
Fun Fact: Direct sum of bialgebras $\neq$ bialgebra
 $\nearrow$ is a weak bialgebra

A weak Hopf algebra over k is a tuple:

$$(H, m: H \otimes H \rightarrow H, u: k \rightarrow H, \Delta: H \rightarrow H \otimes H, \varepsilon: H \rightarrow k, S: H \rightarrow H)$$

so that $(H, m, u, \Delta, \varepsilon)$ is a weak bialgebra over k

& S satisfies weak antipode axioms:



Take a weak ^{Hopf} bialgebra over $\mathbb{k}$, a tuple: $(H, m, u, \Delta, \varepsilon)$ so that

- (H, m, u) is an associative & unital $\mathbb{k}$ -algebra
- (H, Δ, ε) is a coassociative & counital $\mathbb{k}$ -coalgebra
- Δ is multiplicative • u is 'weak comultiplicative' • ε is 'weak multiplicative'
- δ weak antipode

Have special maps:

$$\text{for } \Delta(1) = 1 \otimes 1_2$$

$$\varepsilon_s : H \longrightarrow H$$

$$h \longmapsto 1, \varepsilon(h 1_2)$$

source counital map

$$\varepsilon_t : H \longrightarrow H$$

$$h \longmapsto \varepsilon(1, h) 1_2$$

target counital map

Have special subalgs:

$$H_s = \text{im}(\varepsilon_s)$$

source counital subalg

↙ "bases of H " ↘

$$H_t = \text{im}(\varepsilon_t)$$

target counital subalg

they are separable Frobenius algs

Get a weak ^{Hopf} bialgebra is a ^{Hopf} bialgebra

↑ bases detect this ↑

$$\Leftrightarrow H_s \simeq H_t \simeq \mathbb{k}$$

$$\Leftrightarrow u \text{ is comultiplicative}$$

$$\Leftrightarrow \varepsilon \text{ is multiplicative}$$

Brief History of weak bialgebras

- 1988: Appeared in Ocneanu's study paragroups
- 1991: Weak multiplicativity of ε first appeared in work of Hayashi to understand fusion rules of "Wess-Zumino-Witten models" (2D CFT)
- 1992: Weak comultiplicativity of u first appeared in work of Mack & Schürrer to understand symmetry of low dim'l CFTs.
- (1993, 1996): Hayashi continued to study structures from 1991, introducing "face algs" to study quantum symmetry of certain lattice models & made link to Ocneanu's work
- 1998: Axioms of weak bialgebras were formalized in a preprint by Nill & after in a prepublication by Böhm-Nill-Szlachanyi
 ↓
 "Weak Hopf Algebras I: Integral Theory and C^* -structure"

Post-axiomatization: Weak bialgebras appear in:

- 2001: The Reconstruction Theory for fusion categories
due to Hayashi, Izumi, Szlachany
- 2001: Dynamical Quantum Groups at roots of unity
due to Etingof-Nikshych
- 2003: Modules categories over fusion categories due to Ostrik
- lots of work about structure theory & co/module theory
Nikshych-Vainerman (Böhne) Caenepeel-De Groot Caenepeel-Wang-Yin Böhne-Caenepeel-Janssen

KEY EXAMPLE: GROUPOID ALGEBRA

Take $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ a groupoid $\equiv$ small category where morphisms = isomorphism
 $\uparrow$ set of objects $\nwarrow$ set of maps/arrows

Groupoid Algebra: $(k\mathcal{G}) = \bigoplus_{g \in \mathcal{G}_1} k g$ is a weak Hopf algebra:

$$\mu(g \otimes h) = \begin{cases} gh & \text{if } t(h) = s(g) \\ 0 & \text{else} \end{cases} \quad \eta(1_k) = \sum_{x \in \mathcal{G}_0} \text{id}_x$$

$$\Delta(g) = g \otimes g \quad \varepsilon(g) = 1_k \quad \delta(g) = g^{-1} \quad \forall g \in \mathcal{G}_1$$

$$\text{Here } (k\mathcal{G})_s = (k\mathcal{G})_t = \bigoplus_{x \in \mathcal{G}_0} \text{id}_x$$

KEY EXAMPLE FOR US: HAYASHI'S FACE ALGEBRAS
attached to a finite quiver Q : $\mathcal{H}(Q)$

Take $Q = (Q_0, Q_1, s, t: Q_1 \rightarrow Q_0)$ a finite quiver

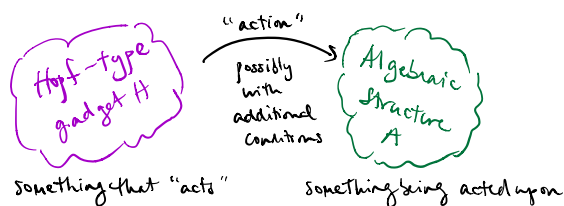
$$\mathcal{H}(Q) = \langle x_{i,j} \mid i,j \in Q_0, p,q \in Q_1 \mid \begin{cases} x_{i,j} x_{k,l} = \delta_{i,k} \delta_{j,l} x_{i,j} & \forall i,j,k,l \in Q_0 \\ x_{s(p),s(q)} x_{p,q} = x_{p,q} = x_{p,q} x_{t(p),t(q)} & \forall p,q \in Q_1 \\ x_{p,q} x_{p',q'} = \delta_{t(p),s(p')} \delta_{t(q),s(q')} x_{p,q} x_{p',q'} & \forall p,q,p',q' \in Q_1 \end{cases} \rangle$$

$$1_{\mathcal{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j} \quad \Delta(x_{a,b}) = \sum_{c \in Q_0} x_{a,c} \otimes x_{c,b} \quad \text{for } a,b \in Q_0$$

$$\varepsilon(x_{a,b}) = \delta_{a,b} 1_K$$

($\mathcal{H}(Q)$ is rarely a weak Hopf algebra)

RECALL THE GAME WE WANT TO PLAY IN ALGEBRAIC QUANTUM SYMMETRY-



Study $\mathcal{H}$ for fixed A

A N -graded locally finite K -algebra

$\mathcal{H}$ coacting linearly

WEAK
QUANTUM
SYMMETRY

noncommutative
&
nonconnected

semi quantum groupoid $\equiv$ weak bialg that coacts

Ex. $A = KQ$, path algebra for a finite quiver $Q = (Q_0, Q_1, s, t: Q_1 \rightarrow Q_0)$

Here, $A_0 = KQ_0$

Ex. $\mathcal{H}(Q)$, weak bialgebra $\equiv$ Hayashi's face algebra attached to Q

COACTIONS OF WEAK BIALGEBRAS

Take H a weak bialgebra. Get monoidal categories of H -comodules

$${}^H\mathcal{M} = (H\text{-comod}, \bar{\otimes}, 1 = H_t, \dots) \quad \& \quad \mathcal{M}^H = (\text{comod-}H, \bar{\otimes}, 1 = H_s, \dots)$$

Here $\bar{\otimes} \neq \otimes_{\mathbb{K}}$ (more complicated)

A $\mathbb{K}$ -algebra A is a left (resp. right) H -comodule algebra if $A \in \text{Alg}({}^H\mathcal{M})$ (resp, $A \in \text{Alg}(\mathcal{M}^H)$)

EXAMPLE $\gamma(Q)$ coacts on $\mathbb{K}Q$ from the left via: & from the right via:

Recall:

$$\gamma(Q) = \langle x_{ij} \mid i, j \in Q_0, \quad p, q \in Q_1 \rangle$$

$$\left(\begin{array}{l} x_{ij} x_{kl} = \delta_{ik} \delta_{jl} x_{ij} \quad \forall i, j, k, l \in Q_0 \\ x_{sp}, x_{tq} x_{p,q} = x_{p,q} x_{s(p), t(q)} \quad \forall p, q \in Q_1 \\ x_{p,q} x_{p',q'} = \delta_{t(p), s(p')} \delta_{t(q), s(q')} x_{p,q} x_{p',q'} \quad \forall p, q, p', q' \in Q_1 \end{array} \right)$$

$$\mathbb{K}Q \xrightarrow{\lambda} \gamma(Q) \otimes \mathbb{K}Q$$

$$e_i \mapsto \sum_{j \in Q_0} x_{ij} \otimes e_j$$

$$p \mapsto \sum_{q \in Q_1} x_{p,q} \otimes q$$

$$\mathbb{K}Q \xrightarrow{\rho} \mathbb{K}Q \otimes \gamma(Q)$$

$$e_i \mapsto \sum_{j \in Q_0} e_j \otimes x_{ji}$$

$$p \mapsto \sum_{q \in Q_1} q \otimes x_{p,q}$$

RECALL Title of Talk: UNIVERSAL QUANTUM SEMIGROUPS

will explain this next

Reviewed this $\equiv$ weak bialgebra coaction

Main theorem: The UNIVERSAL QUANTUM SEMIGROUPS of $\mathbb{K}Q$ is $\gamma(Q)$.

UNIVERSAL QUANTUM (SEMI) GROUPS: crash course

FRAMEWORK:
QUANTUM SYMMETRY

A N -graded locally finite $\mathbb{K}$ -algebra

noncommutative
&
connected

Ex. $\mathbb{K}q[x_1, \dots, x_n]$ for $q \in \mathbb{K}^\times$

H coacting linearly

quantum group $\equiv$ Hopf algebra that coacts

semi quantum group $\equiv$ bialgebra that coacts

Ex. $\mathcal{O}_q(G)$ that coacts

Due to Manin (1988):

A left UQSG of A is a bialgebra $\mathcal{O}^{\text{left}}(A)$ that left coacts on A

→ $\forall$ bialgebra H that left coacts on A

$\exists!$ bialgebra map $\pi: \mathcal{O}^{\text{left}}(A) \rightarrow H$ with:

$$\begin{array}{ccc} A & \xrightarrow{\lambda^0} & \mathcal{O}^{\text{left}}(A) \otimes A \\ \lambda^H \downarrow & \circlearrowleft & \downarrow \pi \otimes \text{id}_A \\ & & H \otimes A \end{array}$$

(like a bialgebra version of an automorphism group)

A right UQSG of A is a bialgebra $\mathcal{O}^{\text{right}}(A)$ that right coacts on A

→ $\forall$ bialgebra H that right coacts on A

$\exists!$ bialgebra map $\pi: \mathcal{O}^{\text{right}}(A) \rightarrow H$ with:

$$\begin{array}{ccc} A & \xrightarrow{\rho^0} & A \otimes \mathcal{O}^{\text{right}}(A) \\ \rho^H \downarrow & \circlearrowleft & \downarrow \text{id} \otimes \pi \\ & & A \otimes H \end{array}$$

$$\begin{array}{ccc} A \xrightarrow{\lambda^0} \mathcal{O}^{\text{left}}(A) \otimes A & & A \xrightarrow{\rho^0} A \otimes \mathcal{O}^{\text{right}}(A) \\ \lambda^H \downarrow \quad \circlearrowleft \quad \downarrow \pi \otimes \text{id}_A & & \rho^H \downarrow \quad \circlearrowleft \quad \downarrow \text{id} \otimes \pi \\ & & H \otimes A \quad \quad A \otimes H \end{array}$$

a certain combination....

A transposed UQSG of A is a bialgebra $\mathcal{O}^{\text{trans}}(A)$ that left & right coacts on A

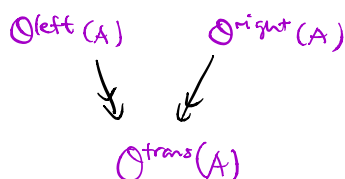
with $(\rho^0)^T: A^* \rightarrow \mathcal{O} \otimes A^*$ equal to λ^0

by identifying basis of A with basis of A^* ... universally

Ex. $A = \mathbb{k}[\sigma_1, \sigma_2]$ rice algebra

$$\begin{array}{ccc} \mathcal{O}^{\text{left}}(A) = \mathbb{k}\langle h_{11}, h_{12}, h_{21}, h_{22} \rangle & \searrow & \mathcal{O}^{\text{right}}(A) = \mathbb{k}\langle h_{11}, h_{12}, h_{21}, h_{22} \rangle \\ \text{not rice algebra} & & \text{rice algebra} \\ \left(\begin{array}{l} h_{11}h_{12} = h_{12}h_{11} \\ h_{21}h_{12} = h_{22}h_{21} \\ [h_{22}, h_{11}] = [h_{21}, h_{12}] \end{array} \right) & & \left(\begin{array}{l} h_{11}h_{21} = h_{21}h_{11} \\ h_{12}h_{22} = h_{22}h_{12} \\ [h_{22}, h_{11}] = [h_{12}, h_{21}] \end{array} \right) \\ & & \text{not rice algebra} \end{array}$$

In general:



Question (Manin) If A is 'nice',
then is $\mathcal{O}^{\text{trans}}(A)$?

UNIVERSAL QUANTUM (SEMI) GROUPS:

FRAMEWORK:
WEAK QUANTUM SYMMETRY

A N -graded locally finite $\mathbb{K}$ -algebra
noncommutative
&
nonconnected
with A_0 commutative & separable

H cofacting linearly
quantum group $\stackrel{\text{old weak}}{\equiv}$ Hopf algebra that cofacts
semi quantum group $\stackrel{\text{old weak}}{\equiv}$ bi algebra that cofacts

Due to HWW (2020)

A left UQSG of A is a ^{weak} bialgebra $\mathcal{O}^{\text{left}}(A)$ that left cofacts on A
with $\mathcal{O}_t \cong A_0$ as left $\mathcal{O}$ -comodule algebras
(target counital subalg) $\swarrow$ (coaction induced by that on A)
left coaction induced by Δ

$\Rightarrow \forall$ ^{weak} bialgebra H that left cofacts on A with $H_t \cong A_0$ as H -comodule algs

$\exists!$ ^{weak} bialgebra map $\pi: \mathcal{O}^{\text{left}}(A) \rightarrow H$ with:

$$\begin{array}{ccc} A & \xrightarrow{\lambda^0} & \mathcal{O}^{\text{left}}(A) \otimes A \\ \lambda^\# \downarrow & \searrow \lambda & \swarrow \pi \otimes \text{id}_A \\ & H \otimes A & \end{array}$$

Need these base preserving conditions
else UQSGd may not exist

Fact: Any nonzero weak bialgebra map $\alpha: H_1 \rightarrow H_2$
preserves counital subalgebras: $(H_1)_t \cong (H_2)_t$.

Right UQSGd & Transposed UQSGds defined likewise

Theorem [HWW 2020]

The UQSGs $\mathcal{O}^{\text{left}}(kQ)$, $\mathcal{O}^{\text{right}}(kQ)$, $\mathcal{O}^{\text{trans}}(kQ)$
are each isomorphic to Hayashi's face algebra $\mathcal{H}(Q)$
as weak bialgebras.

- Speaks to the freeness of $kQ = T_{kQ_0}(kQ_1)$
tensor algebra

Should get $\mathcal{O}^*(\text{free algebra}) = \text{'square of free algebra'}$

$*$ = left, right, trans

Recently,

[Calderín-W].

$\mathcal{H}(Q) = k\hat{Q}$ as k -algs

$\hat{Q}$ = Kronecker square of Q

- Pf/ $\mathcal{O}^{\text{trans}}(kQ) \cong \mathcal{H}(Q)$ was expected

But $\mathcal{O}^{\text{left}}(kQ) \cong \mathcal{H}(Q) \cong \mathcal{O}^{\text{right}}(kQ)$ were tricky!

For graded quotients of path algebras kQ/I :

Prop [HWW, 2020] Take I a graded ideal of kQ .

- Get $\mathcal{O}^{\text{left}}(kQ/I) \cong \mathcal{H}(Q)/\mathcal{I}$ for some biideal $\mathcal{I}$ of $\mathcal{H}(Q)$.
- Similar statements for right & trans UQSGs.

Ex. kQ/I : superpotential algebras, preprojective algebras, ...

For graded quadratic quotients of path algebras kQ/I .

Thm [HWW, 2020] Take I a graded ideal of kQ of degree 2.

• Get $\mathcal{O}^{\text{left}}(kQ/I) \cong \mathcal{O}^{\text{right}}((kQ/I)^{\perp})^{\text{op}}$

↑ & other similar results

↑ quadratic dual $kQ^{\text{op}}/I_{\text{op}}^{\perp}$

Generalization of a result of Manin for kQSGs of connected graded quadratic algs

EX: $\mathcal{O}^{\text{left}}(kQ) \cong \mathcal{O}^{\text{right}}(kQ^{\text{op}}/\langle (kQ^{\text{op}})_2 \rangle)^{\text{op}}$

Studying $\mathcal{O}^*(\underbrace{kQ/I}_{\text{quadratic}})$ now!

Stay tuned

↑ ideal of kQ^{op} generated by all paths of degree 2.

HOPE YOU ENJOYED THE GAME!

Hopf-type gadget H

Something that "acts"

"action"
possibly with additional conditions

Algebraic structure A

Something being acted upon

Thanks for listening!