UNIVERSAL QUANTUM SEMIGROUPOIDS

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/ $\Bbbk$ field.

– IN ALGEBRAIC QUANTUM SYMMETRY –

Hopf-type gadget H —"action"→ Algebraic structure A

possibly with additional conditions

something that "acts" — something being acted upon

GAME I: Study H for fixed A.

GAME II: Fix H then study A.

WE'LL PLAY GAME I for:

All "actions" here will preserve grading of A

A = $\mathbb{N}$-graded, locally finite $\Bbbk$-algebra

$$\left(A = \underbrace{A_0}_{} \oplus A_1 \oplus A_2 \oplus \cdots \ ; \ \dim_{\Bbbk} A_i < \infty \right)$$

more on A_0 later

	A as above	Hopf-type gadget H
CLASSICAL SYMMETRY	Commutative & connected ($A_0 = \Bbbk$)	group (acting by automorphisms) … or more generally … semigroup (acting by endomorphisms)
	Ex. $\Bbbk[x_1, \dots, x_n]$	Ex. $\Bbbk G$, $G \le GL_n(\Bbbk)$ finite that acts –OR– $\Bbbk G$, $G \le Mat_n(\Bbbk)$ finite that acts Dually… $\mathcal{O}(G)$ that coacts

deform

deform

QUANTUM SYMMETRY

"non" = not necessarily

noncommutative* & connected

quantum group ≡ Hopf algebra that co/acts

... or more generally ...

semiquantum group ≡ bialgebra that co/acts

Ex. $\Bbbk_q[x_1, \dots, x_n]$ for $q \in \Bbbk^\times$

Ex. $\mathcal{O}_q(G)$ that coacts

'expand the base'

'expand the base'

WEAK QUANTUM SYMMETRY

noncommutative* & nonconnected*

($\dim_{\Bbbk} A_0$ could be >1)

quantum groupoid ≡ weak Hopf algebra that co/acts

... or more generally ...

semiquantum groupoid ≡ weak bialgebra that co/acts

Ex. $A = \Bbbk Q$, path algebra for a finite quiver $Q = (\underset{\text{finite vertex set}}{Q_0};\ \underset{\text{finite arrow set}}{Q_1};\ \underset{\text{source, target maps}}{s,t : Q_1 \to Q_0})$

Here, $A_0 = \Bbbk Q_0$

Ex. $\mathfrak{h}(Q)$, weak bialgebra ≡ Hayashi's face algebra attached to Q (defined later)

WEAK BI/HOPF ALGEBRAS : crash course

A **weak bialgebra over $\Bbbk$** is a tuple:

$$\Big(\underset{\Bbbk\text{-vs}}{H},\ m: H\otimes H \longrightarrow H,\ u: \Bbbk \longrightarrow H,\ \Delta: H \longrightarrow H\otimes H,\ \varepsilon: H \longrightarrow \Bbbk\Big)$$

$\Bbbk$-linear maps

so that

- (H, m, u) is an associative & unital $\Bbbk$-algebra
- (H, Δ, ε) is a coassociative & counital $\Bbbk$-coalgebra

- Δ is multiplicative

$$\begin{tikzcd}
H\otimes H \arrow[rr, "m"] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[d, "\Delta"] \\
H^{\otimes 4} \arrow[r, "id\otimes flip\otimes id"'] & H^{\otimes 4} \arrow[r, "m\otimes m"'] & H\otimes H
\end{tikzcd}$$

Commutes

- u is 'weak comultiplicative'

$$\begin{tikzcd}
 & \Bbbk \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\
H \arrow[rr, dotted, "\Delta", "\text{if bialgebra}"'] \arrow[d, "\Delta"] & & H\otimes H \arrow[dd, "\Delta\otimes\Delta"] \\
H\otimes H \arrow[d, "\Delta\otimes id"] & & \\
H^{\otimes 3} & & H^{\otimes 4} \arrow[ll, "id\otimes m\otimes id"'] \arrow[dl, "id\otimes flip\otimes id"] \\
 & H^{\otimes 4} \arrow[ul, "id\otimes m\otimes id"] &
\end{tikzcd}$$

Commutes

- ε is 'weak multiplicative'

$$\begin{tikzcd}
 & H^{\otimes 3} \arrow[dl, "id\otimes\Delta^{op}\otimes id"'] \arrow[d, "id\otimes\Delta\otimes id"] \arrow[dr, "m\otimes id"] & \\
H^{\otimes 4} \arrow[d, "m\otimes m"'] & H^{\otimes 4} \arrow[dl, "m\otimes m"] & H\otimes H \arrow[d, "m"] \\
H\otimes H \arrow[rr, dotted, "\text{if bialgebra}", "m"'] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & H \arrow[dl, "\varepsilon"] \\
 & \Bbbk &
\end{tikzcd}$$

Commutes

Fun Fact: Direct sum of bialgebras $\neq$ bialgebra

is a weak bialgebra

A **weak Hopf algebra over $\Bbbk$** is a tuple:

$$\big(H,\ m: H\otimes H \longrightarrow H,\ u: \Bbbk \longrightarrow H,\ \Delta: H \longrightarrow H\otimes H,\ \varepsilon: H \longrightarrow \Bbbk,\ S: H \longrightarrow H\big)$$

so that $(H, m, u, \Delta, \varepsilon)$ is a weak bialgebra over $\Bbbk$

& S satisfies **weak antipode axioms**:

$$\begin{tikzcd}
 & H\otimes H \arrow[rr, "S\otimes id"] & & H\otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, dotted, "\varepsilon", "\text{if Hopf alg}"'] \arrow[d, "id\otimes u"'] & & \Bbbk \arrow[rr, dotted, "u"] & & H \\
H\otimes H \arrow[dr, "id\otimes\Delta"'] & & & & H\otimes H \arrow[u, "id\otimes\varepsilon"'] \\
 & H^{\otimes 3} \arrow[rr, "flip\otimes id"'] & & H^{\otimes 3} \arrow[ur, "id\otimes m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & H\otimes H \arrow[rr, "id\otimes S"] & & H\otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, dotted, "\varepsilon", "\text{if Hopf alg}"'] \arrow[d, "u\otimes id"'] & & \Bbbk \arrow[rr, dotted, "u"] & & H \\
H\otimes H \arrow[dr, "\Delta\otimes id"'] & & & & H\otimes H \arrow[u, "\varepsilon\otimes id"'] \\
 & H^{\otimes 3} \arrow[rr, "id\otimes flip"'] & & H^{\otimes 3} \arrow[ur, "m\otimes id"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[rr, "S"] \arrow[d, "\Delta"'] & & H \\
H\otimes H \arrow[d, "\Delta\otimes id"'] & & H\otimes H \arrow[u, "m"'] \\
H^{\otimes 3} \arrow[rr, "S\otimes id\otimes S"'] & & H^{\otimes 3} \arrow[u, "m\otimes id"']
\end{tikzcd}$$

Take a weak ${}^{\text{Hopf}}$ bialgebra over $\mathbb{k}$, a tuple: $(H, m, u, \Delta, \varepsilon, {}^{S})$ so that

- (H, m, u) is an associative & unital $\mathbb{k}$-algebra
- (H, Δ, ε) is a coassociative & counital $\mathbb{k}$-coalgebra
- Δ is multiplicative • u is 'weak comultiplicative' • ε is 'weak multiplicative'
- S weak antipode

Have special maps : $\varepsilon_s : H \longrightarrow H$, $h \longmapsto 1_1\, \varepsilon(h 1_2)$ — source counital map; $\varepsilon_t : H \longrightarrow H$, $h \longmapsto \varepsilon(1_1 h)\, 1_2$ — target counital map

for $\Delta(1) = 1_1 \otimes 1_2$

Have special subalgs : $H_s = \operatorname{im}(\varepsilon_s)$ — source counital sub alg; $H_t = \operatorname{im}(\varepsilon_t)$ — target counital sub alg

"bases of H"

they are separable Frobenius algs

Get a weak ${}^{\text{Hopf}}$ bialgebra is a ${}^{\text{Hopf}}$ bialgebra $\iff$ $H_s \simeq H_t \simeq \mathbb{k}$

$\iff$ u is comultiplicative

$\iff$ ε is multiplicative

bases detect this

Brief History of weak bialgebras

- 1988 : Appeared in Ocneanu's study paragroups
- 1991 : Weak multiplicativity of ε first appeared in work of Hayashi to understand fusion rules of "Wess-Zumino-Witten models" (2DCFT)
- 1992 : Weak comultiplicativity of u first appeared in work of Mack & Schomerus to understand symmetry of low dim'l CFTs.
- 1993, 1996 : Hayashi continued to study structures from 1991, introducing "face algs" to study quantum symmetry of certain lattice models & made link to Ocneanu's work
- 1998 : Axioms of weak bialgebras were formalized in a preprint by Nill & after in a prepublication by Böhm-Nill-Szlachányi

"Weak Hopf Algebras I: Integral Theory and C^*-structure"

Post-axiomatization: Weak bialgebras appear in:

- 2001: the Reconstruction Theory for fusion categories due to Hayashi, Izumi, Szlachanyi
- 2001: Dynamical Quantum Groups at roots of unity due to Etingof-Nikshych
- 2003: Modules categories over fusion categories due to Ostrik
- lots of work about structure theory & co/module theory

Nikshych-Vainerman, Böhm, Caenepeel-De Groot, Caenepeel-Wang-Yin, Böhm-Caenepeel-Janssen

KEY EXAMPLE: GROUPOID ALGEBRA

Take $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ a groupoid $\equiv$ small category where morphisms = isomorphisms

set of objects; set of maps/arrows

Groupoid Algebra: $\Bbbk\mathcal{G} = \bigoplus_{g \in \mathcal{G}_1} \Bbbk g$ is a weak Hopf algebra:

$$m(g \otimes h) = \begin{cases} gh & \text{if } t(h) = s(g) \\ 0 & \text{else} \end{cases} \qquad u(1_\Bbbk) = \sum_{x \in \mathcal{G}_0} id_x$$

$$\Delta(g) = g \otimes g \qquad \varepsilon(g) = 1_\Bbbk \qquad S(g) = g^{-1} \qquad \forall g \in \mathcal{G}_1$$

Here $(\Bbbk\mathcal{G})_s = (\Bbbk\mathcal{G})_t = \bigoplus_{x \in \mathcal{G}_0} id_x$

KEY EXAMPLE FOR US: HAYASHI'S FACE ALGEBRAS attached to a finite quiver Q : $\mathfrak{H}(Q)$

Take $Q = (Q_0, Q_1, \; s,t : Q_1 \to Q_0)$ a finite quiver

$$\mathfrak{H}(Q) = \frac{\Bbbk\langle x_{i,j}, \; x_{p,q} \mid i,j \in Q_0, \; p,q \in Q_1 \rangle}{\left(\begin{array}{ll} x_{i,j}\, x_{k,l} = \delta_{i,k}\,\delta_{j,l}\, x_{i,j} & \forall i,j,k,l \in Q_0 \\ x_{s(p),s(q)}\, x_{p,q} = x_{p,q} = x_{p,q}\, x_{t(p),t(q)} & \forall p,q \in Q_1 \\ x_{p,q}\, x_{p',q'} = \delta_{t(p),s(p')}\,\delta_{t(q),s(q')}\, x_{p,q}\, x_{p',q'} & \forall p,q,p',q' \in Q_1 \end{array}\right)}$$

$$1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j} \qquad \Delta(x_{a,b}) = \sum_{c \in Q_\ell} x_{a,c} \otimes x_{c,b}$$

$$\varepsilon(x_{a,b}) = \delta_{a,b}\, 1_{\Bbbk} \qquad \text{for } a,b \in Q_\ell$$

($\mathfrak{H}(Q)$ is rarely a weak Hopf algebra)

RECALL THE GAME WE WANT TO PLAY IN ALGEBRAIC QUANTUM SYMMETRY —

Hopf-type gadget H $\xrightarrow{\text{"action"}}$ Algebraic structure A

possibly with additional conditions

something that "acts" — something being acted upon

Study H for fixed A

A: $\mathbb{N}$-graded locally finite $\Bbbk$-algebra

H: co/acting linearly

WEAK QUANTUM SYMMETRY

noncommutative & nonconnected

semiquantum groupoid ≡ weak bialg that co/acts

Ex. $A = \Bbbk Q$, path algebra for a finite quiver $Q = (Q_0; Q_1; s,t : Q_1 \to Q_0)$

Here, $A_0 = \Bbbk Q_0$

Ex. $\mathfrak{H}(Q)$, weak bialgebra ≡ Hayashi's face algebra attached to Q

COACTIONS OF WEAK BIALGEBRAS

Take H a weak bialgebra$/\Bbbk$. Get monoidal categories of H-comodules

$$ {}^H\mathcal{M} = (H\text{-comod}, \overline{\otimes}, \mathbb{1} = H_t, \dots) \;\; \& \;\; \mathcal{M}^H = (\text{comod-}H, \overline{\otimes}, \mathbb{1} = H_s, \dots) $$

Here $\overline{\otimes} \neq \otimes_{\Bbbk}$ (more complicated)

A $\Bbbk$-algebra A is a left (resp. right) H-comodule algebra if $A \in \mathrm{Alg}({}^H\mathcal{M})$
(resp, $A \in \mathrm{Alg}(\mathcal{M}^H)$)

EXAMPLE $\mathcal{Y}(Q)$ coacts on $\Bbbk Q$ from the left via: & from the right via:

Recall:

$$ \mathcal{Y}(Q) = \Bbbk\langle x_{i,j}, x_{p,q} \mid i,j \in Q_0, \; p,q \in Q_1 \rangle \Big/ \left(\begin{array}{ll} x_{i,j}\, x_{k,l} = \delta_{j,k}\,\delta_{j,l}\, x_{i,j} & \forall i,j,k,l \in Q_0 \\ x_{s(p),s(q)}\, x_{p,q} = x_{p,q} = x_{p,q}\, x_{t(p),t(q)} & \forall p,q \in Q_1 \\ x_{p,q}\, x_{p',q'} = \delta_{t(p),s(p')}\,\delta_{t(q),s(q')}\, x_{p,q}\, x_{p',q'} & \forall p,q,p',q' \in Q_1 \end{array} \right) $$

$$ \Bbbk Q \xrightarrow{\lambda} \mathcal{Y}(Q) \otimes \Bbbk Q \qquad\qquad \Bbbk Q \xrightarrow{\rho} \Bbbk Q \otimes \mathcal{Y}(Q) $$

$$ e_i \longmapsto \sum_{j \in Q_0} x_{i,j} \otimes e_j \qquad\qquad e_i \mapsto \sum_{j \in Q_0} e_j \otimes x_{j,i} $$

$$ p \mapsto \sum_{q \in Q_1} x_{p,q} \otimes q \qquad\qquad p \mapsto \sum_{q \in Q_1} q \otimes x_{q,p} $$

RECALL Title of Talk: UNIVERSAL QUANTUM SEMIGROUPOIDS

will explain this next

Reviewed this $\equiv$ weak bialgebra coaction

Main theorem: The UNIVERSAL QUANTUM SEMIGROUPOID of $\Bbbk Q$ is $\mathcal{Y}(Q)$.

UNIVERSAL QUANTUM (SEMI) GROUPS: crash course

FRAMEWORK: QUANTUM SYMMETRY

A $\mathbb{N}$-graded locally finite $\Bbbk$-algebra

noncommutative & connected

Ex. $\Bbbk_q[x_1, \dots, x_n]$ for $q \in \Bbbk^\times$

H coacting linearly

quantum group $\equiv$ Hopf algebra that coacts

semiquantum group $\equiv$ bialgebra that coacts

Ex. $\mathcal{O}_q(G)$ that coacts

Due to Manin (1988):

A left UQSG of A is a bialgebra $\mathcal{O}^{left}(A)$ that left coacts on A

s.t. $\forall$ bialgebra H that left coacts on A

$\exists!$ bialgebra map $\pi : \mathcal{O}^{left}(A) \longrightarrow H$ with:

$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathcal{O}}"] \arrow[dr, "\lambda^H"'] & \mathcal{O}^{left}(A) \otimes A \arrow[d, dashed, "\pi \otimes id_A"] \\ & H \otimes A \end{tikzcd}$$

(like a bialgebra version of an automorphism group)

A right UQSG of A is a bialgebra $\mathcal{O}^{right}(A)$ that right coacts on A

s.t. $\forall$ bialgebra H that right coacts on A

$\exists!$ bialgebra map $\pi : \mathcal{O}^{right}(A) \longrightarrow H$ with:

$$\begin{tikzcd} A \arrow[r, "\rho^{\mathcal{O}}"] \arrow[dr, "\rho^H"'] & A \otimes \mathcal{O}^{right}(A) \arrow[d, dashed, "id \otimes \pi"] \\ & A \otimes H \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathcal{O}}"] \arrow[dr, "\lambda^H"'] & \mathcal{O}^{left}(A) \otimes A \arrow[d, dashed, "\pi \otimes id_A"] \\ & H \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\rho^{\mathcal{O}}"] \arrow[dr, "\rho^H"'] & A \otimes \mathcal{O}^{right}(A) \arrow[d, dashed, "id \otimes \pi"] \\ & A \otimes H \end{tikzcd}$$

a certain combination....

A transposed UQSG of A is a bialgebra $\mathcal{O}^{trans}(A)$ that left & right coacts on A

with $(\rho^{\mathcal{O}})^T : A^* \longrightarrow \mathcal{O} \otimes A^*$ equal to $\lambda^{\mathcal{O}}$

by identifying basis of A with basis of A^* ... universally

Ex. $A = \Bbbk[v_1, v_2]$ nice algebra

$\mathcal{O}^{left}(A) = \Bbbk\langle h_{11}, h_{12}, h_{21}, h_{22} \rangle \Big/ \left(\begin{array}{l} h_{11}h_{12} = h_{12}h_{11} \\ h_{21}h_{22} = h_{22}h_{21} \\ [h_{22}, h_{11}] = [h_{21}, h_{12}] \end{array} \right)$ not nice algebra

$\mathcal{O}^{trans}(A) = \Bbbk[h_{ij}]_{i,j=1}^{2}$ nice algebra

$\mathcal{O}^{right}(A) = \Bbbk\langle h_{11}, h_{12}, h_{21}, h_{22} \rangle \Big/ \left(\begin{array}{l} h_{11}h_{21} = h_{21}h_{11} \\ h_{12}h_{22} = h_{22}h_{12} \\ [h_{22}, h_{11}] = [h_{12}, h_{21}] \end{array} \right)$ not nice algebra

In general:

$$\begin{tikzcd} \mathcal{O}^{left}(A) \arrow[dr] & & \mathcal{O}^{right}(A) \arrow[dl] \\ & \mathcal{O}^{trans}(A) & \end{tikzcd}$$

Question (Manin) If A is 'nice', then is $\mathcal{O}^{trans}(A)$?

UNIVERSAL QUANTUM (SEMI) GROUPOIDS:

FRAMEWORK: WEAK QUANTUM SYMMETRY

A $\mathbb{N}$-graded locally finite $\Bbbk$-algebra

noncommutative & nonconnected with A_0 commutative & separable

H coacting linearly

quantum group(oid) = (weak) Hopf algebra that coacts

semiquantum group(oid) = (weak) bialgebra that coacts

Due to HWWW (2020)

A left UQSGd of A is a weak bialgebra $\mathcal{O}^{left}(A)$ that left coacts on A

with $\mathcal{O}_t \cong A_0$ as left $\mathcal{O}$-comodule algebras

(target counital subalg) left coaction induced by Δ — coaction induced by that on A

$\ni$. $\forall$ weak bialgebra H that left coacts on A with $H_t \cong A_0$ as H-comodule algs

$\exists!$ weak bialgebra map $\pi: \mathcal{O}^{left}(A) \longrightarrow H$ with:

$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathcal{O}}"] \arrow[dr, "\lambda^H"'] & \mathcal{O}^{left}(A) \otimes A \arrow[d, dashed, "\pi \otimes id_A"] \\ & H \otimes A \end{tikzcd}$$

Need these 'base preserving' conditions else UQSGd may not exist

Fact: Any nonzero weak bialgebra map $\alpha: H_1 \longrightarrow H_2$ preserves counital subalgebras: $(H_1)_t \cong (H_2)_t$.

Right UQSGd & Transposed UQSGds defined likewise

Theorem [HWWW 2020]

The WQSGds $\mathcal{O}^{left}(\Bbbk Q)$, $\mathcal{O}^{right}(\Bbbk Q)$, $\mathcal{O}^{trans}(\Bbbk Q)$ are each isomorphic to Hayashi's face algebra $\mathfrak{H}(Q)$ as weak bialgebras.

- Speaks to the freeness of $\Bbbk Q \simeq \underbrace{T_{\Bbbk Q_0}(\Bbbk Q_1)}_{\text{tensor algebra}}$

  Should get $\mathcal{O}^{*}(\text{free algebra}) =$ 'square of free algebra'

  $* =$ left, right, trans

  Recently, [Calderón-W].
  $\mathfrak{H}(Q) = \Bbbk\hat{Q}$ as $\Bbbk$-algs
  $\hat{Q}$ = Kronecker square of Q

- Pf/ $\mathcal{O}^{trans}(\Bbbk Q) \cong \mathfrak{H}(Q)$ was expected

  But $\mathcal{O}^{left}(\Bbbk Q) \cong \mathfrak{H}(Q) \cong \mathcal{O}^{right}(\Bbbk Q)$ were tricky!

For graded quotients of path algebras $\Bbbk Q/I$:

Prop [HWWW, 2020] Take I a graded ideal of $\Bbbk Q$.

- Get $\mathcal{O}^{left}(\Bbbk Q/I) \cong \mathfrak{H}(Q)/\tilde{I}$ for some biideal $\tilde{I}$ of $\mathfrak{H}(Q)$.
- similiar statements for right & trans WQSGds.

Ex. $\Bbbk Q/I$: superpotential algebras, preprojective algebras, ...

For graded quadratic quotients of path algebras $\Bbbk Q/I$.

Thm [HWWW, 2020] Take I a graded ideal of $\Bbbk Q$ of degree 2.

- Get $\mathcal{O}^{left}(\Bbbk Q/I) \cong \mathcal{O}^{right}\big(\underbrace{(\Bbbk Q/I)^!}_{\text{quadratic dual } \Bbbk Q^{op}/I^{\perp}_{op}}\big)^{op}$

& other similar results

Generalization of a result of Manin for UQSGs of connected graded quadratic algs

EX: $\mathcal{O}^{left}(\Bbbk Q) \cong \mathcal{O}^{right}\Big(\Bbbk Q^{op} / \underbrace{\langle (\Bbbk Q^{op})_2 \rangle}_{\text{ideal of } \Bbbk Q^{op} \text{ generated by all paths of degree 2.}}\Big)^{op}$

Studying $\mathcal{O}^*(\underbrace{\Bbbk Q/I}_{\text{quadratic}})$ now!

Stay tuned

HOPE YOU ENJOYED THE GAME!

Hopf-type gadget H $\xrightarrow{\text{"action"}}$ Algebraic structure A

possibly with additional conditions

something that "acts" — something being acted upon

Thanks for listening!