On Degenerations and Deformations of Sklyanin Algebras

by

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Like many doctorate theses of others, this one is not perfect. Please note that the material in Chapters 3 and 4 has been published, but:

(1) There is a corrigendum to "Degenerate Sklyanin algebras and Generalized Twisted Homogeneous Coordinate rings", J. Algebra 322 (2009) 2508-2527, regarding Section 3.2.2.
(2) There is a minor error in the appendix (Proposition A.1).
(3) The material in Chapter 5 is not published.

Please feel free to contact me at notlaw@temple.edu if you have any questions or concerns, especially if you would like to continue the work in Chapter 5.

-Chelsea Walton, Temple University

For my family and friends.

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Introduction

1.1 Overview

Noncommutative rings and their representations have long had a presence in mathematics, at least since the birth of quantum mechanics in the 1930s gave rise to the Weyl algebra. To study these objects, we naturally first use algebraic techniques, yet when these methods fail, one must reach outside the world of algebra. Recently the development of *noncommutative projective algebraic geometry* studies noncommutative rings and their representations by exploiting the interplay of commutative graded rings and projective geometric objects in classical algebraic geometry. This thesis contributes to this enterprise by enlarging the class of noncommutative rings which have been analyzed in this fashion.

Let k denote an algebraically closed field of characteristic not equal to 2 or 3. A k-algebra A is **connected graded** if $A = \bigoplus_{i \in \mathbb{N}} A_i$ is $\mathbb{N}$-graded with $A_0 = k$. Furthermore we always consider algebras that are finitely generated in degree one. To exhibit classes of noncommutative graded algebras that are analyzed with techniques of noncommutative projective algebraic geometry, we provide a few historical remarks. In the mid-1980s, Michael Artin and William Schelter launched noncommutative projective algebraic geometry in their attempt to complete the following project.

Classify the noncommutative graded analogues of

the polynomial ring in three variables.

This task proved difficult as the ring-theoretic and homological behavior of these rings could not be determined using purely algebraic techniques. First it is necessary to clarify what is meant by a noncommutative analogue of $k[x, y, z]$. Artin and Schelter settled on the following definition.

Definition I.1. [AS87] A connected graded ring A is called a **three-dimensional Artin-Schelter (AS) regular algebra** if A satisfies the following three properties:

(i) global dimension 3;

(ii) finite Gelfand Kirillov dimension; and

(iii) the AS-Gorenstein condition: $\mathrm{Ext}^i_A(k, A) = \delta_{i,3} \cdot k$.

A few years later, the classification of three-dimensional AS-regular algebras was achieved in the seminal paper of Michael Artin, John Tate, and Michel van den Bergh [ATVdB90]. The toughest challenge was the investigation of the following family of algebras.

Definition I.2. The **three-dimensional Sklyanin algebras**, denoted by $S(a, b, c)$ or $Skly_3$, are generated by three noncommuting variables x,y,z, subject to three relations:

$$\begin{aligned} ayz + bzy + cx^2 &= 0 \\ azx + bxz + cy^2 &= 0 \\ axy + byx + cz^2 &= 0 \end{aligned} \tag{1.1}$$

for $[a : b : c] \in \mathbb{P}^2_k \smallsetminus \mathfrak{D}$ where

$$\mathfrak{D} = \{[0:0:1], [0:1:0], [1:0:0]\} \ \cup \ \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

This definition is more general than the standard definition given in [ATVdB90]. Namely the latter is associated to the geometry of a smooth cubic curve in $\mathbb{P}^2$, whereas our version is associated to either $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$ (Theorem I.4).

We now provide a brief description of Artin-Tate-van den Bergh's geometric approach to understanding noncommutative graded rings. To contrast with the noncommutative setting, note that the geometric object typically associated to a commutative graded ring is a projective scheme Proj, the set of all homogeneous prime two-sided ideals except the irrelevant ideal. One associates to the polynomial ring in two variables $k[x,y]$, for example, the projective line $\text{Proj}(k[x,y]){:=}\mathbb{P}^1$. However, noncommutative graded rings do not have many two-sided ideals in general. For instance, let R_q be the ring generated by noncommuting variables x, y, subject to the relation $xy = qyx$ for q nonzero, not a root of unity (a noncommutative analogue of $k[x,y]$). Then $\text{Proj}(R_q)$ consists of merely three non-irrelevant prime ideals! Unless one wants to consider a space of three points corresponding to the ring R_q, an alternative geometry is required.

Hence given a noncommutative ring A, rather than employing ideals, the points of the corresponding geometric object are thought of module-theoretically. More explicitly, a noncommutative point is associated to the following A-module, which behaves like a homogeneous coordinate ring of a closed point in the commutative setting.

Definition I.3. A **point module** over a graded ring A is a cyclic graded left A-module $M = \bigoplus_{i\geq 0} M_i$ where $\dim_k M_i = 1$ for all $i \geq 0$.

The parameterization of point modules leads us to a noncommutative generalization of the commutative object Proj. Namely for an arbitrary connected graded algebra A, the parameterization of isomorphism classes of A-point modules (if it exists as a projective scheme) is called the **point scheme** X of A; see Definition II.6. In fact, the point scheme of R_q is $\mathbb{P}^1$, yet such a predictable parameterization does not always occur; cf. Theorem I.4. If R is a commutative graded ring generated in degree 1, then the point scheme of R is simply $\mathrm{Proj}(R)$.

Returning to algebra, we build a (noncommutative) graded ring B corresponding to the point scheme X of A. Furthermore this ring B is often used to study the ring-theoretic behavior of A as we will see later. Now to build B, take a projective scheme X, with invertible sheaf $\mathcal{L}$ and automorphism σ on X. Then one can form the **twisted homogeneous coordinate ring** $B = B(X, \mathcal{L}, \sigma)$ as defined in Definition II.9. In fact when $\sigma = \mathrm{id}_X$, then we get the commutative *section ring* $B(X, \mathcal{L})$ of X with respect to $\mathcal{L}$, a ring that is typically used to understand (coherent sheaves on) projective schemes (Theorem II.1).

Thus point schemes and twisted homogeneous coordinate rings are respectively genuine noncommutative analogues of projective schemes Proj and section rings $B(X, \mathcal{L})$ in classical algebraic geometry.

Now in the theorem below, Artin-Tate-van den Bergh use these constructions to provide a geometric framework specifically associated to the Sklyanin algebras.

Theorem I.4. *[ATVdB90] (i) The point scheme of $S = S(a,b,c)$ is isomorphic to:*

$$E = E_{abc} : \mathbb{V}\left((a^3+b^3+c^3)xyz - (abc)(x^3+y^3+z^3)\right) \overset{i}{\subset} \mathbb{P}^2. \tag{1.2}$$

Here $E = \mathbb{P}^2$ if one of the a, b, or c is 0 and the sum of the other two parameters is 0; E is a smooth cubic curve if $abc \neq 0$ and $(3abc)^3 \neq (a^3+b^3+c^3)^3$; and E is a singular cubic curve otherwise.

(ii) Assume that E is smooth. Given the invertible sheaf $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^2}(1)$ on E, and a certain automorphism σ of E (which corresponds to the shift functor on point mod-

ules), we have that the twisted homogeneous coordinate ring $B = B(E, i^*\mathcal{O}_{\mathbb{P}^2}(1), \sigma)$ *is a noetherian domain. Moreover* $dim_k B_0 = 1$*, and for* $d \geq 1$*:* $dim_k B_d = 3d$ *if* E *is a smooth cubic curve; and* $dim_k B_d = dim_k\ k[x,y,z]_d$ *if* $E = \mathbb{P}^2$*.*

(iii) If $E \neq \mathbb{P}^2$*, then there exists a regular normal element* $g \in S$*, homogeneous of degree 3, so that* $B \cong S/Sg$ *as graded rings. Moreover if* E *is also smooth, then* $Skly_3$ *is also a noetherian domain, with the same Hilbert series as the polynomial ring in three variables; namely* $H_S(t) = (1-t)^{-3}$*.* □

Thus part (iii) implies that twisted homogeneous coordinate rings play a crucial role in determining the ring-theoretic and homological behavior of the corresponding Sklyanin algebras. The aim of this dissertation is to exploit and expand the above geometric techniques to study variants of Sklyanin algebras and other noncommutative connected graded rings.

We now list the contents of each chapter in more depth. Chapter 2 contains the background material for this dissertation. Chapter 3 analyzes degenerations of Sklyanin algebras with use of generalized twisted homogeneous coordinate rings; introductory remarks are provided in §1.2. Chapters 4 and 5 investigate the representation theory of ungraded deformations of Sklyanin algebras; introduction given in §1.3. Computational results and computer routines are presented in Chapter 4 and in the appendix; a synopsis of these results is provided in §1.4.

1.2 Generalizing twisted homogeneous coordinate rings

During the last few years, a number of examples of (even noetherian) algebras have appeared for which the techniques of [ATVdB90, ATVdB91] are inapplicable. This is because the point modules for these algebras cannot by parameterized by a projective scheme of finite type (see for example [KRS05]). Consequently one cannot form a corresponding twisted homogeneous coordinate ring. In Chapter 3, we explore a recipe suggested in [ATVdB90, §3] for building a generalized twisted homogeneous coordinate ring for *any* connected graded ring. In particular, we provide a geometric approach to examine the following degenerations of Sklyanin algebras. Moreover the results listed in this section appear in [Wal09].

Definition I.5. The rings $S(a,b,c)$ from Definition I.2 with $[a:b:c] \in \mathfrak{D}$ are called **degenerate Sklyanin algebras**, denoted by S_{deg}.

The name is motivated by the fact that the geometry of S_{deg} involves a degenerate cubic curve. On the other hand, we establish algebraic properties of S_{deg} in the following theorem.

Theorem I.6. *(Lemma III.8, Corollary III.9, Propositions III.11 and III.12) The degenerate three-dimensional Sklyanin algebras have Hilbert series* $H_{S_{deg}}(t) = (1+t)(1-2t)^{-1}$*. They are neither left or right noetherian, nor are they domains. Moreover* S_{deg} *is Koszul and has both infinite global dimension and infinite Gelfand Kirillov dimension. We also have that* $Z(S_{deg}) = k$*.*

By this result, we see that degenerate Sklyanin algebras possess entirely different ring-theoretic characteristics than Sklyanin algebras. However we can still construct geometric data for these algebras with the methods of [ATVdB90]. More precisely, we make use of the point modules over S_{deg} to construct a generalization of a point scheme for S_{deg}. Although this geometric data is not of finite type, the geometry of degenerate Sklyanin algebras is remarkably nice and still describes their class of point modules. We provide the details of this phenomenon in the theorem below.

Definition-Lemma I.7. *[ATVdB90] Let A be a connected graded ring generated by $n+1$ elements of degree one.*

1. *A* **truncated point module of length d** *over A is a cyclic graded left A-module $M = \bigoplus_{i\geq 0} M_i$ where $dim_k M_i = 1$ for $0 \leq i \leq d$ and $dim_k M_i = 0$ for $i > d$.*
2. *The d^{th}* **truncated point scheme** *V_d parameterizes isomorphism classes of length d truncated point modules.*
3. *Assume that the inverse limit of $\{V_d\}$, with respect to restrictions of projection maps onto the first $d-1$ coordinates of $(\mathbb{P}^n)^{\times d}$, stabilizes to a projective scheme. This inverse limit is referred to as the* **point scheme** *of A, and it parameterizes A-point modules.*

Theorem I.8. *(Proposition III.26) For $d \geq 2$, the truncated point schemes $V_d \subset (\mathbb{P}^2)^{\times d}$ corresponding to S_{deg} are isomorphic to either:*
(i) a union of three copies of $(\mathbb{P}^1)^{\times \frac{d-1}{2}}$ and three copies of $(\mathbb{P}^1)^{\times \frac{d+1}{2}}$, if d is odd, or;
(ii) a union of six copies of $(\mathbb{P}^1)^{\times \frac{d}{2}}$, if d is even.

As previously mentioned, the inverse limit of the truncated point schemes, V_d, of degenerate Sklyanin algebras do not stabilize to produce a projective scheme of finite type. Hence we cannot mimic the approach of [ATVdB90] to construct a twisted homogeneous coordinate ring associated to S_{deg}. So to build a graded ring associated to the geometry of S_{deg} we instead make use of the V_d, or rather the **point scheme data** of S_{deg}, and a method from [ATVdB90, 3.17] that supplies us with the recipe to construct the following geometric ring.

Definition I.9. [ATVdB90, §3] The **point parameter ring** $P = \bigoplus_{d\geq 0} P_d$ is an $\mathbb{N}$-graded, associative ring corresponding to the subschemes V_d of $(\mathbb{P}^2)^{\times d}$ (Definition

I.7). Here $P_d = H^0(V_d, \mathcal{L}_d)$ where $\mathcal{L}_d$ is the restriction of the invertible sheaf

$$pr_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \ldots \otimes pr_d^* \mathcal{O}_{\mathbb{P}^2}(1)$$

to V_d. The multiplication map $P_i \times P_j \to P_{i+j}$ is defined by taking global sections of the isomorphism $pr^*_{1,\ldots,i}(\mathcal{L}_i) \otimes_{\mathcal{O}_{V_{i+j}}} pr^*_{i+1,\ldots,i+j}(\mathcal{L}_j) \xrightarrow{\sim} \mathcal{L}_{i+j}$.

Point parameter rings have rarely appeared in the literature since their origin in [ATVdB91], so these rings are not understood in general. The next results of this thesis describe the behavior of the first non-trivial examples of these rings. As a consequence, we extend the class of noncommutative algebras that are traditionally studied in noncommutative projective algebraic geometry to *all* connected graded algebras (for which a point scheme does not necessarily exist).

Theorem I.10. *(Proposition III.35, Theorem III.36, Corollary III.40) The point parameter ring $P = P(S_{deg})$ is generated in degree one. Since $(S_{deg})_1 \cong (P(S_{deg}))_1$, we have that P is a factor of the corresponding S_{deg}. Furthermore it has Hilbert series $H_P(t) = (1+t^2)(1+2t)[(1-2t^2)(1-t)]^{-1}$.*

Thus Theorem I.10 yields a result surprisingly similar to Theorem I.4 (pertaining to the ring surjection from $Skly_3$ onto $B(E)$), despite the fact that by Theorem I.6, the rings $Skly_3$ and S_{deg} are completely unalike.

Corollary I.11. *(Corollary III.41) The ring $P = P(S_{deg})$ has exponential growth. Moreover P is neither right noetherian, Koszul, nor a domain.*

Therefore the behavior of $P(S_{deg})$ resembles that of S_{deg} and it is natural to ask if other noncommutative algebras can be analyzed in a similar fashion. Some of these further directions are discussed in §3.4.

Section 3.5 in particular discusses the investigation of the rings S_{deg} and $P(S_{deg})$ via Piontkovskii and Polishchuk's recent work on *coherent noncommutative algebraic* *geometry* [Pio08, Pol05]. Their results thus far include the classification of noncommutative projective lines and the definition of a coordinate ring of a noncommutative coherent projective scheme. Now if S_{deg} and $P(S_{deg})$ are coherent, then the rings will contribute to the theory on *coherent noncommutative projective schemes*. In Proposition III.55, we show that the degenerate Sklyanin algebras are indeed coherent. However, we do not know if this is true for $P(S_{deg})$.

1.3 Representations of deformed Sklyanin algebras

Traditional techniques of noncommutative projective algebraic geometry analyze (noncommutative) graded rings with projective geometric data. Chapter 4 considers the question of how one might study ungraded variants of these algebras. Specifically, we examine the following ungraded deformations of three-dimensional Sklyanin algebras.

Definition I.12. For $i = 1, 2, 3$, let a, b, c, d_i, e_i be scalars in k with $[a : b : c] \notin \mathfrak{D}$ (Definition I.2). The **deformed Sklyanin algebra**, S_{def}, is generated by three noncommuting variables x, y, z, subject to three relations:

$$ayz + bzy + cx^2 + d_1x + e_1 = 0$$

$$azx + bxz + cy^2 + d_2y + e_2 = 0$$

$$axy + byx + cz^2 + d_3z + e_3 = 0.$$

One can show that for (almost all) parameters (a, b, c), the ring S_{def} is a PBW-deformation of $Skly_3$ [BT07][EG, §3]. The study of these rings is physically motivated as they first appeared in the study of vacua in string theory. More precisely, the analysis of m-dimensional simple modules over S_{def} is relevant to the study of *supersymmetric Yang-Mills theories* with m D-branes [BJL00]. We now discuss significant results towards achieving the following aim:

Classify the simple finite-dimensional modules over the algebra S_{def}***.***

Our approach is as follows. First, homogenize the relations of S_{def} with a central element w. Then, study the representation theory of the resulting *central extension* D of $Skly_3$. Since simple finite-dimensional D-modules are precisely those over $Skly_3$ or S_{def} (Lemma II.32), our objective can be reformulated in terms of studying the (not necessarily graded) simple finite-dimensional modules over the graded algebras, $Skly_3$ and D. The latter are nice objects in the sense of [Sta02], as they are three- and four-dimensional AS-regular algebras respectively.

Results on the classification of simple finite-dimensional modules over $Skly_3$ are provided in Chapter 4, and these are summarized in Theorem I.13 below. First we consider the following properties of $Skly_3$. Note that $Skly_3$ satisfies a *polynomial identity* or is *PI* [MR01, Chapter 13] precisely when $|\sigma| = n < \infty$ (Theorem I.18, [For74, ST94]). This scenario is of interest in representation theory as there are many finite-dimensional simple modules over such rings [BG97, §3]. Recall that $S = Skly_3$ contains a regular central element g, homogeneous of degree 3, so that $S/Sg \cong B$ a twisted homogeneous coordinate ring (Theorem I.4). Moreover B is PI if and only if S is PI.

Theorem I.13. *(Propositions IV.13, IV.16, IV.18, IV.19)*

(i) In the case of $|\sigma| = \infty$*, the only finite-dimensional simple module of* $Skly_3$ *is the trivial module.*

(ii) Assume that $|\sigma| < \infty$*. Let* M *be a finite-dimensional simple module over* $Skly_3$*.*

(a) If M *is* g*-torsionfree, then* $\dim_k M = |\sigma|$ *when* $(3, |\sigma|) = 1$*;*

and $|\sigma|/3 \leq \dim_k M \leq |\sigma|$ *when* $(3, |\sigma|) \neq 1$.

(b) Otherwise if M *is* g*-torsion, then* $\dim_k M = |\sigma|$.

In fact, we will see in Proposition I.21 that $\mathrm{PIdeg}(S) = \mathrm{PIdeg}(B) = |\sigma|$ when $|\sigma| < \infty$.

Therefore to study the representation theory of S_{def}, we now analyze the representation theory of D. This is done in Chapter 5. Fortunately, the noncommutative geometry of D has been examined in [LBSVdB96]. It is known that the point scheme P_D of D is the union of the point scheme of $Skly_3$ (the object E from Theorem I.4) and a finite set of points $\{s_i\}$. The automorphism σ_D on P_D is given by σ on E and the identity on the finite set of points. Therefore $|\sigma_D| = |\sigma|$, and conjectures pertaining to the aforementioned research objective are formed involving this order. We claim the following dichotomy for D.

Conjecture I.14. *(Conjecture V.9) The central extension D is either PI or all simple finite-dimensional D-modules are 1-dimensional.*

In fact, we can describe the class of simple 1-dimensional D-modules.

Lemma I.15. *(Lemma V.5, Corollary V.7) The nontrivial 1-dimensional simple D-modules are simple quotients of the point modules:*

$$\left\{ \frac{D}{Dy_1 + Dy_2 + Dy_3} \mid \mathbb{V}(y_1, y_2, y_3) = s_i \in P_D \right\}.$$

The set $\mathcal{S} = \{s_i\}$ is described in the cases of physical significance in the appendix.

The approach to verifying Conjecture I.14 first involves the analysis of **(fat) point modules**, which by definition are 1-critical graded modules (of multiplicity greater than 1) over D. (These differ slightly from the commutative version.)

Lemma I.16. *[LS93] [SS93] Let M be a simple finite-dimensional D-module. Then M is a quotient of some 1-critical graded module. Moreover if M arises as the quotient of a fat point module, then* $\dim_k M > 1$.

Hence given a central extension D of $Skly_3$, we aim to show that either fat point modules of D do not exist or D is PI. Now we proceed by studying D according to

$|\sigma_D| = |\sigma|$. If $|\sigma| = \infty$, then D is not PI (Proposition V.10). For such a D, we study its fat point modules in §5.4, which we conjecture do not exist.

On the other hand, our claim pertaining to the PI property for the central extension D of $Skly_3$ is as follows. If $|\sigma| = 1, 2$, then we claim that the center of D is generated by only two elements of low degree, so D would not be PI in this case (Conjecture V.12, V.13). In fact in §5.4, we also analyze the fat point modules of D for the $|\sigma| = 2$ case. On the other hand if $|\sigma| = 3, 6$, then we claim that D is PI (Conjecture V.15, Remark V.16); evidence is provided in §5.3. In summary:

Conjecture I.17. *For d_i and e_i generic, D is not PI when $|\sigma| = 1, 2$, and is PI when $|\sigma| = 3$ or 6.*

Thus in contrast with Sklyanin algebras (i.e. Theorem I.18 below), we believe that the relationship between order of the automorphism σ_D and the structure of the center of D is quite subtle.

1.4 Computational results on $Skly_3$

Certain results presented in Chapter 4 and in the appendix deal with computational aspects of the three-dimensional Sklyanin algebras. Recall that $Skly_3 = S(a, b, c)$ comes equipped with geometric data: E, a cubic curve or $\mathbb{P}^2$, with automorphism σ; and these are defined by the parameters a,b,c. For the result of this section we assume that E is smooth, equivalently we have that either: one of a, b, c equals 0 with the sum of the other two parameters equal to 0; or $abc \neq 0$ and $(a^3 + b^3 + c^3)^3 \neq (3abc)^3$. We determine the role of these parameters in governing the behavior of the Sklyanin algebra, work motivated by the following crucial result of [ATVdB91]:

Theorem I.18. *[ATVdB91, Theorem 7.1] The algebra $Skly_3 = S(a,b,c)$ is module finite over its center if and only if $|\sigma| < \infty$.*

In fact, the first condition is equivalent to S being PI [For74, ST94]. Hence if the conditions of the above theorem hold, then the behavior of $S(a,b,c)$ is intimately tied to the structure of its center; how 'close the tie' is made precise by the *PI degree*. Furthermore recall that the twisted homogeneous coordinate ring $B(E,\mathcal{L},\sigma)$ arises as a homomorphic image of S (Theorem I.4), so B is PI when S is PI. This naturally gives rise to the following problems.

Questions I.19. .

1. *Fix $n \in \mathbb{N}$. Let σ_{abc} be the automorphism of the point scheme E_{abc} introduced in Theorem I.4. Classify parameters (a,b,c) for which σ_{abc} has order n.*

2. *Given (a,b,c) with $|\sigma_{abc}| < \infty$, determine the PI degree of the rings $S(a,b,c)$ and $B(E_{abc}, i^*\mathcal{O}_{\mathbb{P}^2}(1), \sigma_{abc})$.*

The first problem has been settled for Sklyanin algebras associated to elliptic curves over $\mathbb{Q}$ [LB94]. For an arbitrary field k, however, the question remains open. Partial progress is reported in Proposition A.1 in the appendix. More precisely, we have the following result.

Proposition I.20. *Given $n = 1,\dots,6$, we know the parameters (a,b,c) of S for which $|\sigma_{abc}| = n$.*

On the other hand, the result below completes the second task.

Proposition I.21. *(Corollaries IV.15 and IV.19) Provided parameters a,b,c of $Skly_3 = S(a,b,c)$ for which $|\sigma_{abc}| < \infty$, both the PI degree of $S(a,b,c)$ and the PI degree of $B(E_{abc}, i^*\mathcal{O}_{\mathbb{P}^2}(1), \sigma_{abc})$ are equal to $|\sigma_{abc}|$.*

CHAPTER II

Background Material

This chapter discusses the background material for the study of the noncommutative geometry of degenerate Sklyanin algebras, and the representation theory of deformed Sklyanin algebras. In §2.1 we introduce noncommutative projective algebraic geometry (NCPAG), a field launched by the study of noncommutative graded rings via techniques of classical algebraic geometry. The next section introduces the notion of Gelfand-Kirillov dimension, which pertains to the growth of the algebras and of the modules considered in this thesis. Section 2.3 discusses the representation theory of (not necessarily graded) noncommutative rings with methods from NCPAG. Lastly, §§2.4-2.5 are dedicated to the theory behind polynomial identity rings and Bergman's diamond lemma respectively.

Notation. Fix a $\mathbb{Z}$-graded ring A generated in degree one. As presented in [AZ94, §2], we consider the following module categories:

A-**Mod**, the category of left A-modules;

A-**mod**, the category of noetherian left A-modules, which is simply the category of finitely generated left A-modules if A is noetherian;

A-**Gr**, the category of $\mathbb{Z}$-graded left A-modules with degree preserving homomorphisms;

A-gr, the subcategory of A-Gr consisting of noetherian graded left A-modules.

A module $M \in A$-Gr is **right bounded** if $M_i = 0$ for all $i >> 0$. Moreover M is **torsion** if all $m \in M$, we have that $A_{\geq t} \cdot m = 0$ for some t. This is equivalent to the condition that M is the (not necessarily finite) sum of right bounded modules. Now we form the following categories:

Tors(A), the full subcategory of A-Gr consisting of torsion graded left A-modules;

tors(A), Tors(A) $\cap$ A-gr, which is the category of finite dimensional graded A-modules if A is $\mathbb{N}$-graded.

We can also form the quotient categories:

A-**QGr** $= A$-Gr/Tors(A);

A-**qgr** $= A$-gr/tors(A).

The objects of the second category, A-qgr, consist of equivalence classes $[M]$ for $M \in A$-gr. Namely we have that $M \sim N$ if there exists an integer n for which $M_{\geq n} \cong N_{\geq n}$, where $M_{\geq n} = \bigoplus_{d \geq n} M_d$.

Furthermore for a projective variety X, we also consider **(q)coh(X)**, the category of (quasi-)coherent sheaves on X.

2.1 Noncommutative projective algebraic geometry (NCPAG)

By no means is this a complete introduction to noncommutative projective algebraic geometry; see the survey article of Stafford and van den Bergh [SvdB01] for a more detailed discussion. The paper [AZ94] is the standard reference to the categorical approach to this field.

2.1.1 Commutative motivation

The field of noncommutative projective algebraic geometry is motivated by techniques of classical projective algebraic geometry. Namely we develop noncommutative analogues of projective schemes and corresponding homogeneous coordinate rings to study noncommutative graded rings that are suspected to behave like commutative graded rings. For instance such methods can be used to determine whether a noncommutative graded ring $A = \bigoplus_{i \in \mathbb{N}} A_i$ has the Hilbert series as that of a polynomial ring $k[x_1, \ldots, x_n]$, to say:

$$H_A(t) := \sum_{i \in \mathbb{N}} \dim_k A_i \cdot t^i \;\; = \;\; (1-t)^{-n}.$$

Let us recall some results of classical projective algebraic geometry used to study commutative graded algebras. The remaining material of this subsection is summarized from [Har77, §II.5]. First we construct projective geometric data from a given algebra, then we use this data to build a geometric ring. Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be an $\mathbb{N}$–graded commutative algebra, generated in degree 1 over $R_0 = k$. We first associate to R the geometric object $\mathrm{Proj}(R)$, which by definition is the set of all homogeneous prime ideals $\mathfrak{p}$ except the irrelevant ideal $R_+ := \bigoplus_{i \geq 1} R_i$. In particular, the (not necessarily closed) points of the *projective scheme* $X = \mathrm{Proj}(R)$ are in 1-1 correspondence with such $\mathfrak{p}$. However noncommutative rings do not possess many two-sided ideals in general, and since we want these methods to generalize, we instead consider the 1-1 correspondence between points of X and quotients of R by these prime ideals $\mathfrak{p}$. Specifically, each closed point of X corresponds to a homogeneous prime ideal $\mathfrak{p}$ whose quotient, $R/\mathfrak{p}$, is a cyclic graded R-module with Hilbert series $H_{k[x]}(t) = \sum_{i=0}^{\infty} t^i$.

Next we consider an invertible sheaf $\mathcal{L}$ of X and construct the *section ring* $B = B(X, \mathcal{L}) := \bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{\otimes d})$. Here B has the natural multiplication $B_d \times B_e \to B_{d+e}$

induced from taking global sections of isomorphisms: $\mathcal{L}^{\otimes d} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e} \cong \mathcal{L}^{\otimes d+e}$. When $(X, \mathcal{L})$ are nice (e.g. irreducible and normal, and ample respectively), then B is a well behaved ring (e.g. a Noetherian domain) [Gro61, Chapter 2], [Smi97, §1].

How does this help us analyze our ring R? This is answered by the following theorem of Serre.

Theorem II.1. *[Ser55, Proposition 7.8] (Serre's Theorem)*

(i) Let R be a commutative graded ring, finitely generated in degree one over a field R_0. Let X = Proj(R). Then the functor R-gr $\to$ coh(X) sending M to $\tilde{M}$, induces an equivalence of categories: R-qgr $\sim$ coh(X).

(ii) Conversely, let $B = B(X, \mathcal{L})$ where $\mathcal{L}$ is an ample invertible sheaf on X. Then the map $\mathcal{M} \mapsto \Gamma_*(\mathcal{M}) = \bigoplus_{d \geq 0} H^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})$ induces an equivalence of categories: coh$(X) \sim B$-qgr.

(iii) If R defined as in part (i) and $B = B(ProjR, \mathcal{O}_{ProjR}(1))$, then $R_d = B_d$ for $d >> 0$. Hence we have that R-qgr $\sim$ B-qgr.

Therefore we have that R and B in the theorem above reflect each other ring-theoretically. In fact if R is normal, then R is isomorphic to B. This discussion and an example are depicted in the figures below.

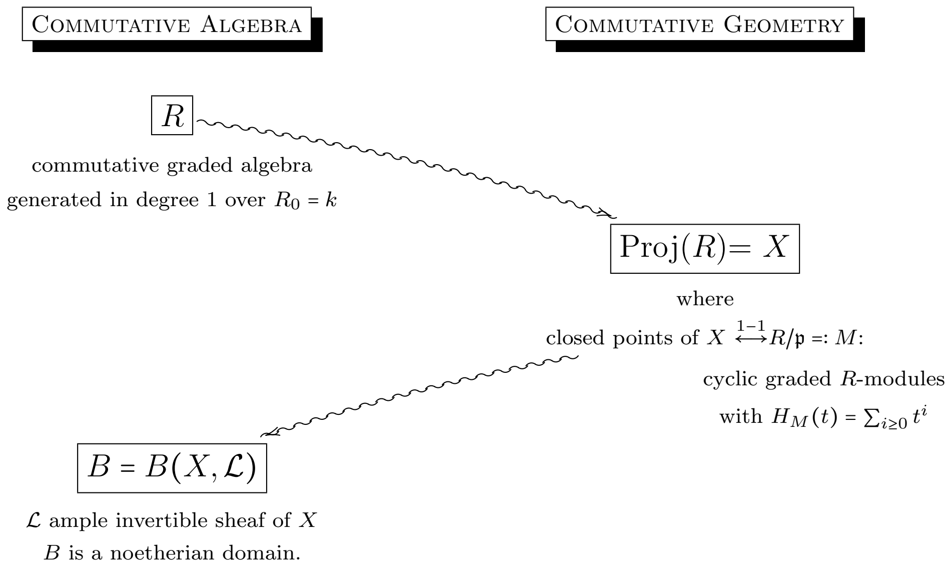

Figure 2.1: Interactions between commutative graded algebras and commutative projective geometry

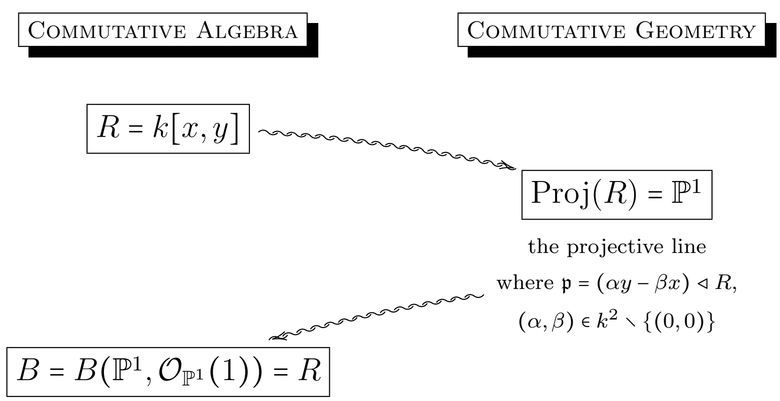

Figure 2.2: Example of the discussion in §2.1.1.

2.1.2 Techniques of Artin-Tate-van den Bergh (ATV)

We now discuss techniques of Artin, Tate, and van den Bergh to generalize the results of §2.1.1 to the noncommutative setting. Let us consider a noncommutative $\mathbb{N}$–graded ring $A = \bigoplus_{i \in \mathbb{N}} A_i$, generated in degree 1 over $A_0 = k$. To study this ring geometrically, then we need to make sense of $\mathrm{Proj}(A)$ in the noncommutative setting.

As mentioned in Chapter 1, if $A = R_q = k\{x,y\}/(xy - qyx)$ where $q \neq 0, \sqrt{1}$, then A has only three two-sided homogeneous non-irrelevant prime ideals. Instead we consider the modules given in Figure 2.1, and associate to A a geometric object X whose closed points are in 1-1 correspondence with the following modules.

Definition II.2. A cyclic graded left A–module M with Hilbert series $H_M(t) = \sum_{i=0}^{\infty} t^i$ is called a **point module** of A.

Loosely speaking if X exists, then it is a commutative projective scheme which is referred to as the *point scheme* of A (see Definition-Lemma II.6 for the formal statement).

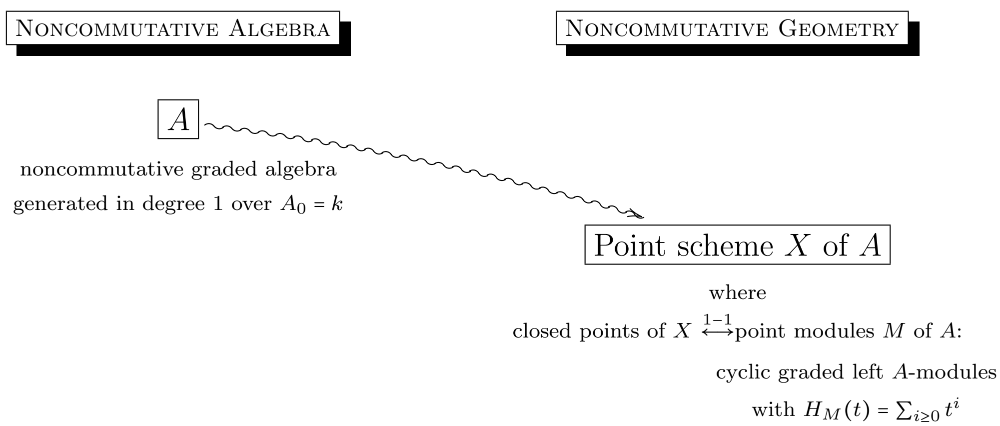

Figure 2.3: ATV technique: Algebra to Geometry

Since point modules are generally difficult to understand, we first parameterize the following modules.

Definition II.3. A **truncated point module of A of length d** is a cyclic graded left A-module M with Hilbert series $H_M(t) = \sum_{i=0}^{d-1} t^i$.

The parameterization of truncated point modules is achieved by the following method.

Step 1: Present A as $T(V)/I$ where V is an $n+1$ dimensional k-vector space, $T(V)$ is the free algebra $k\{x_0, \dots, x_n\}$, and I is the two-sided ideal of relations.

Step 2: Given a relation f of degree d, we have that $f \in I_d \subseteq V^{\otimes d} \cong ((V^*)^{\otimes d})^*$, which induces the linear functional $\tilde{f} : (V^*)^{\times d} \to k$.

Step 3: Next form truncated point schemes:

Definition II.4. The d^{th} **truncated point scheme** of A is

$$V_d := \{(p_1, \dots, p_d) \in \mathbb{P}(V^*)^{\times d} \mid \tilde{f}(p_1, \dots, p_d) = 0, \forall f \in I_d\},$$

which lies in $(\mathbb{P}^n)^{\times d}$. Observe that V_d is the set of zeros of multilinearizations of the degree d relations of A.

Step 4: Finally we have the parameterization of truncated point modules of A via the next result.

Lemma II.5. *[ATVdB90, Proposition 3.9] The scheme V_d parameterizes the set of isomorphism classes of truncated point modules of length d. More precisely, V_d represents the functor of flat families of such modules.*

Proof sketch. We verify the bijective correspondence. Given a point $(p_1, \dots, p_d) \in V_d$ where $p_i = [\lambda_{0i} : \lambda_{1i} : \dots : \lambda_{ni}]$, we can build a truncated module of length d, $M = \bigoplus_{j=0}^{d-1} k \cdot m_j$, where m_j is the generator of the j^{th} graded piece of M. Here the A-action of M is: $x_i * m_j = \lambda_{ij} \cdot m_{j+1}$. Conversely when given such a truncated module, the scalars λ_{ij} from its A-action yield a point in V_d. $\square$

Now we can parameterize the set of isomorphism classes of point modules.

Notation. Let $\pi_d : (\mathbb{P}^n)^{\times d+1} \to (\mathbb{P}^n)^{\times d}$ be projection onto the first d coordinates of $(\mathbb{P}^n)^{\times d+1}$. Note that $V_d \subseteq (\mathbb{P}^n)^{\times d}$ and that $\pi_d(V_{d+1}) \subseteq V_d$ according to [ATVdB90, Equation (3.18)]. We also refer to the induced map $V_{d+1} \to V_d$ by π_d.

Definition-Lemma II.6. *[ATVdB90, Corollary 3.13] The pairs* $(V_d, \pi_d : V_{d+1} \to V_d)$ *form an inverse system of schemes. Note that if one of the projections* π_d *is an isomorphism, then the inverse system* $\{V_n\}$ *is constant for* $n \geq d$*. Hence* $\varprojlim V_n \cong V_d$ *and we denote this limit by* X*. We call* X *the* **point scheme** *of* A*.*

In general we refer to the family $\{(V_d, \pi_d)\}_{d\geq 0}$ *as the* **point scheme data** *of* A*.*

More generally we can consider the pro-scheme structure of $\varprojlim V_d$, though this limit may not have carry the structure of a projective scheme. Refer to [NSb] for details of its realization of as an *algebraic stack*.

Let us illustrate some examples of point scheme data.

Examples II.7. (a) Let $A = k\{x, y, z\}$, the free algebra. Since there are no relations of A, we have that V_d is $(\mathbb{P}^2)^{\times d}$. The point scheme data does not stabilize to form a projective scheme, yet note that $\varprojlim V_d \cong (\mathbb{P}^2)^{\times\infty}$. Hence the highly noncommutative algebra $k\{x, y, z\}$ is associated to the large geometric projective object $(\mathbb{P}^2)^{\times\infty}$.

(b) For the other extreme, consider $A = k[x, y, z]$, the polynomial algebra. We have that V_d is isomorphic to the diagonal $\{(p, \ldots, p) \in (\mathbb{P}^2)^{\times d} \mid p \in \mathbb{P}^2\}$. A computation of V_2 is given below. Now we see that the point scheme of A is isomorphic to $\mathbb{P}^2$.

(b') We explicitly show how V_2 is the set of zeros of multilinearizations of degree 2 relations of $A = k[x_0, x_1]$. To begin, note that

$$V_2 = \{([\lambda_{01} : \lambda_{11}], [\lambda_{02} : \lambda_{12}]) \in \mathbb{P}^1_{[x_{01}:x_{11}]} \times \mathbb{P}^1_{[x_{02}:x_{12}]} \mid \tilde{f}([\lambda_{01} : \lambda_{11}], [\lambda_{02} : \lambda_{12}]) = 0,\ \forall f \in I_2\}.$$

Observe that the only relation of A of degree two is

$$f = x_0x_1 - x_1x_0 \subseteq V \otimes V := (k \cdot x_{01} \oplus k \cdot x_{11}) \otimes (k \cdot x_{02} \oplus k \cdot x_{12}).$$

Thus $\tilde{f} : V^* \times V^* \to k$ is defined by $\tilde{f}(\phi) = \phi(f) := x_{01}x_{12} - x_{11}x_{02}$. By the construction of V_2, we have that $\lambda_{01}\lambda_{12} - \lambda_{11}\lambda_{02} = 0$. In other words, $[\lambda_{01} : \lambda_{11}] = [\lambda_{02} : \lambda_{12}]$ and $V_2 = \{(p, p) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid p \in \mathbb{P}^1\}$ as desired.

(c) The point scheme of a commutative ring R generated in degree one is isomorphic to $\mathrm{Proj}(R)$. Hence point schemes are genuine noncommutative analogues of commutative projective schemes.

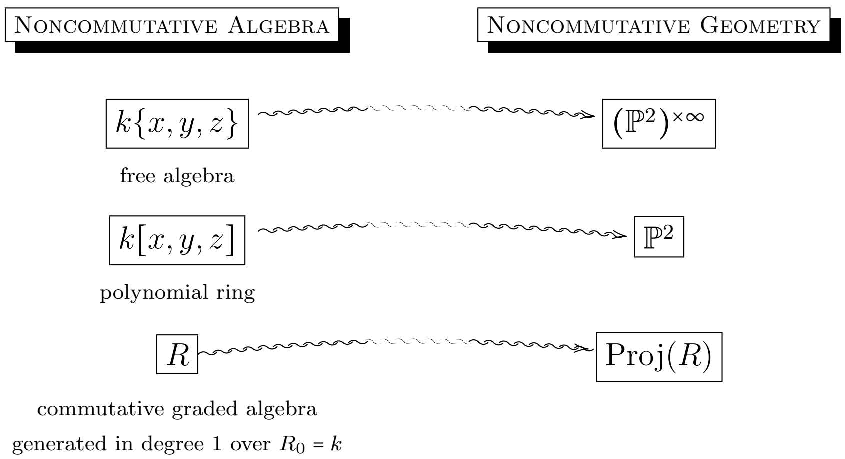

Figure 2.4: Examples of ATV technique: Algebra to Geometry

Another piece of geometric data that we frequently employ is the following automorphism σ of the point scheme X.

Definition II.8. Assuming existence, let X be the point scheme of a connected graded k-algebra A. For a point $p \in X$, let $M(p)$ be a corresponding A-point module. We say that an automorphism σ of X is **induced by the shift functor on point modules of** A if $M(p)[1]_{\geq 1} = M(\sigma^{-1}p)$ for all $p \in X$.

All automorphisms of point schemes in this thesis are induced by the shift functor on point modules; see [ATVdB91, §6] for more details about this notion.

Now we return from noncommutative geometry back to noncommutative algebra, under the vital condition that $\varprojlim V_d$ stabilizes as a projective scheme (to X). We can then construct the coordinate ring below.

Notation. Given a projective scheme X, an invertible sheaf $\mathcal{L}$ on X, and $\sigma \in \mathrm{Aut}(X)$, we write $\mathcal{L}^\sigma$ for the pullback of $\mathcal{L}$ along σ. For any open set U of X, we have that $\mathcal{L}^\sigma(U) = \mathcal{L}(\sigma U)$.

Definition II.9. [ATVdB90, §6] Given a point scheme X as above, let $\mathcal{L}$ be an invertible sheaf on X, and let $\sigma \in \mathrm{Aut} X$. The **twisted homogeneous coordinate ring** $B = B(X, \mathcal{L}, \sigma)$ of X with respect to $\mathcal{L}$ and σ is an $\mathbb{N}$-graded ring:

$$B = \bigoplus_{d \in \mathbb{N}} H^0(X, \mathcal{L}_d)$$

where $\mathcal{L}_0 = \mathcal{O}_X$, $\mathcal{L}_1 = \mathcal{L}$, and $\mathcal{L}_d = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\sigma \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \mathcal{L}^{\sigma^{d-1}}$ for $d \geq 2$. Here multiplication is given by the natural map:

$$\begin{aligned} B_d \otimes B_e &\cong H^0(X, \mathcal{L}_d) \otimes_k H^0(X, \mathcal{L}_e) \\ &\cong H^0(X, \mathcal{L}_d) \otimes_k H^0(X, \mathcal{L}_e^{\sigma^d}) \xrightarrow{\phi} H^0(X, \mathcal{L}_{d+e}) \cong B_{d+e} \end{aligned}$$

where ϕ is obtained by taking global sections of the isomorphism $\mathcal{L}_d \otimes_{\mathcal{O}_X} \mathcal{L}_e^{\sigma^d} \cong \mathcal{L}_{d+e}$.

Observe that when $\sigma = \mathrm{id}|_X$, then $B(X, \mathcal{L}, \sigma)$ is the commutative section ring $B(X, \mathcal{L})$ from §2.1.1 (see Figure 2.1). We illustrate in the following figure the interaction between noncommutative algebra and geometry in the fashion of Artin-Tate-van den Bergh as described above; an example is then provided.

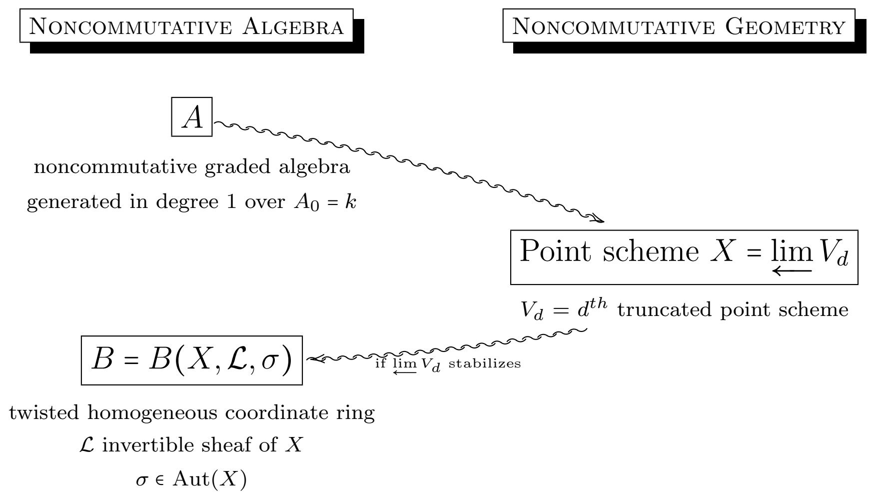

Figure 2.5: Interactions between noncommutative algebra and noncommutative geometry

Now consider the ring $R_q = k\{x,y\}/(yx - qxy)$ where $q \neq 0, \sqrt{1}$, which is suspected to behave like a polynomial ring in two variables.

Lemma II.10. *There is a 1-1 correspondence between closed points of $\mathbb{P}^1$ and isomorphism classes of point modules over R_q. Hence the point scheme of R_q is $\mathbb{P}^1$.*

Proof. First note that the Hilbert series $H_{R_q}(t)$ is equal to $(1-t)^{-2}$.

Take a closed point $[\alpha : \beta] \in \mathbb{P}^1$. Here $[\alpha : \beta]$ represents an equivalence class of tuples $(p_1, p_2) \in k^2 \smallsetminus \{(0,0)\}$ with $(p_1, p_2) = (\lambda\alpha, \lambda\beta)$ for some nonzero $\lambda \in k$. The point $[\alpha : \beta]$ is in bijective correspondence with the left ideal $R_q(\alpha y - \beta x)$ of R_q. Now the cyclic graded left R_q-module $M = R_q/R_q(\alpha y - \beta x)$ is an R_q-point module as

$$H_M(t) \;=\; H_{R_q}(t) - H_{R_q(\alpha y - \beta x)}(t) \;=\; H_{R_q}(t) - tH_{R_q}(t) \;=\; (1-t)^{-1}.$$

Conversely let M be a point module over R_q with generator m. Then $\mathrm{ann}_{R_q}(m) = \{r \in R_q \mid rm = 0\}$ is a left ideal of R_q, and $M \cong R_q/\mathrm{ann}_{R_q}(m)$. Since $H_M(t) = (1-t)^{-1}$ and $H_{R_q}(t) = (1-t)^{-2}$, we have that $H_{\mathrm{ann}_{R_q}(m)}(t) = t(1-t)^{-2}$. Hence $\mathrm{ann}_{R_q}(m)$ is of the form $R_q a$ where $a \in (R_q)_1$. In other words, $\mathrm{ann}_{R_q}(m) = R_q(\alpha y - \beta x)$ for some

$(\alpha, \beta) \in k^2 \smallsetminus \{(0,0)\}$. Moreover M is isomorphic to another R_q-point module M' with generator m' if and only if $\mathrm{ann}_{R_q}(m) = \mathrm{ann}_{R_q}(m')$. Here by the same reasoning $\mathrm{ann}_{R_q}(m') = R_q a' = R_q(\alpha' y - \beta' x)$ for some $a' \in (R_q)_1$ and $(\alpha', \beta') \in k^2 \smallsetminus \{(0,0)\}$. Hence $M \cong M'$ if and only if $a = \lambda a'$ for some nonzero $\lambda \in k$. Therefore the module M corresponds to the closed point $[\alpha : \beta] \in \mathbb{P}^1$. □

Lemma II.11. *Given the projective line $\mathbb{P}^1$ with invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$ and automorphism $\sigma[x : y] = [x : qy]$, we have that the twisted homogeneous coordinate ring $B(\mathbb{P}^1, \mathcal{L}, \sigma)$ is isomorphic to the ring R_q.*

Proof. We refer the reader to [SvdB01, Example 3.4]. □

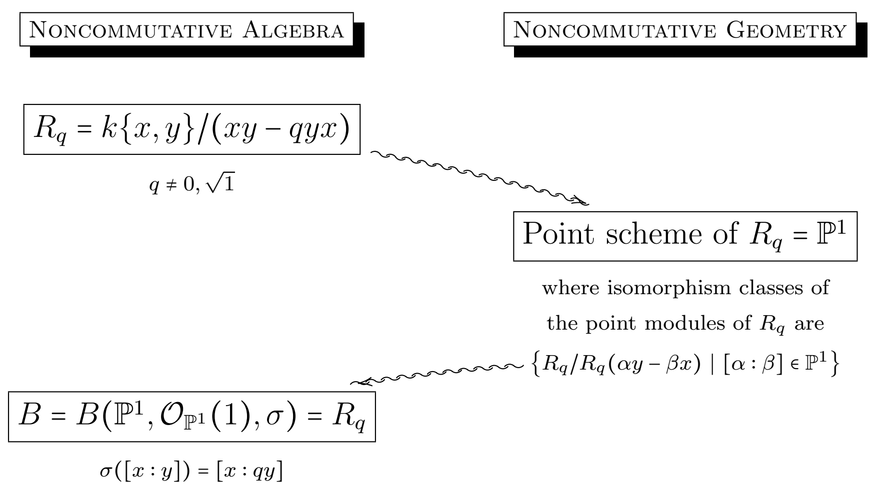

Figure 2.6: Example of the discussion in §2.1.2.

Observe that the example above did not require the computation of truncated point modules; this is rarely the case.

Though tedious, constructing twisted homogeneous coordinate rings is worthwhile due to the following remarkable results of Artin-van den Bergh and Keeler. First we consider some terminology.

Definition II.12. [AVdB90] Given a projective scheme X, an invertible sheaf $\mathcal{L}$ on X, and $\sigma \in \mathrm{Aut}(X)$, then $\mathcal{L}$ is **right σ-ample** if for all $\mathcal{F} \in \mathrm{coh}(X)$ and $n >> 0$:

(a) $\mathcal{F} \otimes \mathcal{L}_n$ is generated by global sections where $\mathcal{L}_n = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\sigma} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{L}^{\sigma^{n-1}}$;

(b) $H^q(X, \mathcal{F} \otimes \mathcal{L}_n) = 0$ for all $q > 0$.

We say that $\mathcal{L}$ is **left σ-ample** if (a) and (b) hold for the sheaf $\mathcal{F}^{\sigma^n} \otimes \mathcal{L}_n$.

Theorem II.13. *[Kee00, Corollary 5.1] The right σ-ample and left σ-ample conditions are equivalent.*

The concept of σ-ampleness is rather delicate [Kee00], but for the work in this thesis, we point out that invertible sheaves on irreducible curves and on projective spaces are ample if and only if they are σ-ample [Kee00, Proposition 5.6, Corollary 5.4] [Sta].

Theorem II.14. *[AVdB90] (Noncommutative Serre's Theorem) If X is an irreducible projective scheme, $\mathcal{L}$ is a σ- ample invertible sheaf, then the twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$ is a Noetherian domain. Secondly we have an equivalence of categories: B-qgr $\sim$ coh(X).*

We can often relate our connected graded ring A to a twisted homogeneous coordinate ring B through a ring surjection $A \twoheadrightarrow B$, and in this case the nice properties of B (discussed in Theorem II.14) lift to A. This is extremely useful when the ring-theoretic behavior of A cannot be determined using purely algebraic techniques; an example of such an algebra A is provided in the next subsection.

2.1.3 Analysis of Sklyanin algebras via Artin-Tate-van den Bergh methods

Here we apply methods of Artin-Tate-van den Bergh to study three-dimensional Sklyanin algebras (Definition I.2), first computing their point schemes..

Proposition II.15. *[ATVdB90, §4] The point scheme X (Definition-Lemma II.6) of $S = Skly_3$ is isomorphic to the object $E_{abc} \overset{i}{\subset} \mathbb{P}^2$ from Equation (1.2):*

$$E = E_{abc} : \mathbb{V}\left((a^3+b^3+c^3)xyz - (abc)(x^3+y^3+z^3)\right) \subset \mathbb{P}^2,$$

where $E = \mathbb{P}^2$ or a cubic curve. In particular, we have the following statements. Recall that V_d denotes the d^{th} truncated point scheme of S (Definition II.4).

(i) For $d \geq 2$, the morphism $\pi_d : V_{d+1} \to V_d$ is an isomorphism of varieties.

(ii) The morphism $\pi_1 : V_2 \to \mathbb{P}^2$ is an isomorphism onto its image, which is the object E.

(iii) Let $\pi_1' : V_2 \to \mathbb{P}^2$ be the morphism induced by projection onto the second copy of $\mathbb{P}^2$. Then π_2' is an isomorphism onto its image, which again is E above.

(iv) The morphism $\sigma = \pi_1' \circ \pi_1^{-1} : E \to E$ is an automorphism of E. The closed subvariety V_2 of $\mathbb{P}^2 \times \mathbb{P}^2$ then is isomorphic to the graph of σ.

Proof. We borrow the presentation of this proof from [Ste, Proposition 5.8.3].

(i) We must show that a point $p = (p_1, \ldots, p_d) \in V_d$ extends uniquely to a point $p' = (p_1, \ldots, p_d, p_{d+1}) \in V_{d+1}$ and that the morphism sending p to p' is a morphism of varieties. Consider multilinearizations of the three relations of $Skly_3$:

$$f_i := ay_iz_{i+1} + bz_iy_{i+1} + cx_ix_{i+1},$$

$$g_i := az_ix_{i+1} + bx_iz_{i+1} + cy_iy_{i+1},$$

$$h_i := ax_iy_{i+1} + by_ix_{i+1} + cz_iz_{i+1}.$$

Note that $V_d = \mathbb{V}(f_i, g_i, h_i)_{1 \leq i \leq d-1} \subseteq (\mathbb{P}^2)^{\times d}$ for $d \geq 2$ with $V_1 \cong \mathbb{P}^2$. Take $\mathtt{M_i}$ to be the $3{\times}3$ matrix:

$$\begin{pmatrix} cx_i & bz_i & ay_i \\ az_i & cy_i & bx_i \\ by_i & ax_i & cz_i \end{pmatrix}$$

and note that $f_i = g_i = h_i = 0$ is equivalent to:

$$\mathtt{M_i} \cdot (x_{i+1}, y_{i+1}, z_{i+1})^T = 0. \tag{2.1}$$

Moreover $f_i = g_i = h_i = 0$ is also equivalent to:

$$(x_i, y_i, z_i) \cdot \mathtt{M_{i+1}} = 0. \tag{2.2}$$

To show the uniqueness of the morphism sending p to p', we must verify that the solution $(x_{d+1}, y_{d+1}, z_{d+1})$ of Equation (2.1) is projectively unique. In other words, the matrix $\mathtt{M_d}$ must have rank 2 at every point of V_d. Since Equation (2.2) holds for $i = d-1$, we know that rank($\mathtt{M_d}$)≤ 2. Suppose that rank($\mathtt{M_d}$)=1, i.e. every 2×2 minor of $\mathtt{M_d}$ vanishes. This yields the following set of equations:

$$\begin{gathered} c^2x_dy_d = abz_d^2, \quad b^2y_dz_d = acx_d^2, \quad a^2x_dz_d = bcy_d^2, \\ a^2y_dz_d = bcx_d^2, \quad c^2x_dz_d = aby_d^2, \quad b^2x_dy_d = acz_d^2, \\ b^2x_dz_d = acy_d^2, \quad a^2x_dy_d = bcz_d^2, \quad c^2y_dz_d = abx_d^2. \end{gathered}$$

We leave it to the reader to show that these equations imply that the point $[a:b:c]$ is in the set $\mathfrak{D}$ of Definition I.2, a contradiction. Thus rank($\mathtt{M_d}$) $=2$ for every point of V_d. Moreover since the coordinates of p_{d+1} are given locally by 2×2 minors of $\mathtt{M_d}$, we have that π_d is an isomorphism of varieties.

(ii) The image of $\pi_1 : V_2 \to \mathbb{P}^2$ is the set of all points $[x_1 : y_1 : z_1] \in \mathbb{P}^2$ so that there exists $[x_2 : y_2 : z_2] \in \mathbb{P}^2$ with Equation (2.1) satisfied for $i = 1$. This is equivalent to

$$\{[x_1 : y_1 : z_1] \in \mathbb{P}^2 \mid \det(\mathtt{M_1}) = 0\}.$$

Hence $\mathrm{im}(\pi_1)$ is the closed subset $E_{abc} \subseteq \mathbb{P}^2$ above. Furthermore π_1 is an isomorphism onto E as rank($\mathtt{M_1}$)=2 for every point $[x_1 : y_1 : z_1] \in E$. The function $\pi_1^{-1} : E \to V_2$ is a morphism of varieties as it is defined locally by the 2×2 minors of $\mathtt{M_1}$.

(iii) This is symmetric to the proof of (ii).

(iv) [ATVdB90, Proposition 3.7] By the definition of σ, we have that

$$V_2 = \{(p, \sigma(p)) \mid p \in E\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$$

Thus V_2 is isomorphic to the graph of σ. $\square$

Note that $E = \mathbb{P}^2$ if one of a, b, c equals 0 and the sum of the other two parameters equals 0; E is an elliptic curve if $abc \neq 0$ and $(a^3 + b^3 + c^3)^3 \neq (3abc)^3$; and E is a singular cubic curve otherwise.

Next for E smooth, we form the twisted homogeneous coordinate ring $B = B(E, i^*\mathcal{O}_{\mathbb{P}^2}(1), \sigma)$ where σ is given in Proposition II.15. We have by Theorem II.14 that B is a Noetherian domain. Moreover if E is a smooth cubic curve, then $\dim_k B_d = 3d$ for $d \geq 1$ [ATVdB90, Theorem 6.6(ii)]. If $E = \mathbb{P}^2$, then $\dim_k B_d = \dim_k [x, y, z]_d$ for $d \geq 0$. To yield results about $Skly_3$, we employ the following result of Artin-Tate-van den Bergh.

Theorem II.16. *[ATVdB90, Theorem 6.8] Assume that the point scheme E_{abc} of $S(a, b, c)$ is smooth. If $E = \mathbb{P}^2$, then $S(a, b, c)$ is isomorphic to a twisted homogeneous coordinate ring $B(\mathbb{P}^2, \mathcal{L}, \sigma)$. If E is an elliptic curve, then there is a ring surjection from $S(a, b, c)$ to the twisted homogeneous coordinate ring $B = B(E_{abc}, i^*\mathcal{O}_{\mathbb{P}^2}(1), \sigma)$, with kernel generated by a degree 3 central, regular element (often denoted by g).*

Remark II.17. In fact, the element g is explicitly given in [AS87, (10.17)]. It was computed with Affine, a noncommutative package for the computer algebra system Maxima. For $S(a, b, c)$, we have that:

$$g = c(c^3 - b^3)y^3 + b(c^3 - a^3)yxz + a(b^3 - c^3)xyz + c(a^3 - c^3)x^3.$$

Now we have the result below, which forms the heart of [ATVdB90].

Corollary II.18. *[ATVdB90, Theorem 8.1][RZ08, proof of Corollary 4.6(2)] The three-dimensional Sklyanin algebras $Skly_3$, with a smooth point scheme E_{abc}, are Noetherian domains of polynomial growth; indeed their Hilbert series are $H_S(t) = (1-t)^{-3}$.*

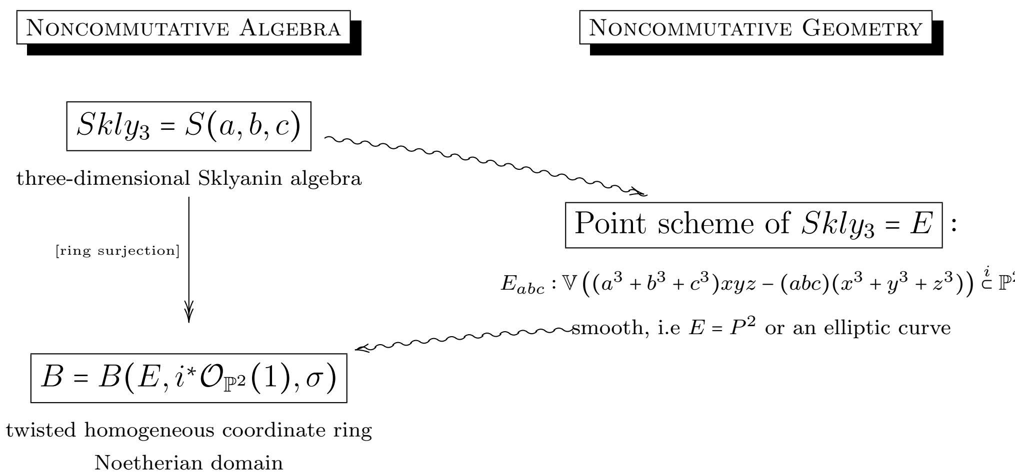

Figure 2.7: ATV technique applied to $Skly_3$

2.1.4 Linear noncommutative geometry of higher dimensions

We introduced in §2.1.2 the notion of a point module and of a point scheme associated to a connected graded k-algebra A. Likewise we can define higher dimensional noncommutative geometric objects corresponding to A, objects which are particularly useful in Chapter 5.

Definition II.19. A cyclic graded left A-module M with Hilbert series $H_M(t) = (1-t)^{-(d+1)}$ is called a d-**linear module**. In particular, d-linear modules with $d = 0, 1, 2$ are **point modules**, **line modules**, and **plane modules** respectively.

We can also discuss the parameterizations of isomorphism classes of such modules; these are referred to as d-**linear schemes**. We point out that the $d = 1$ schemes, or rather **line schemes**, have been studied in several articles including [ATVdB91, LBSVdB96, LS93, SV02a, SV02b].

Continuing with our analysis of three-dimensional Sklyanin algebras, we describe their line and plane schemes below.

Example II.20. [ATVdB91] The line modules of $S = Skly_3$ are precisely modules of the form S/Sv where v is a nonzero element of S_1. These modules are in bijective correspondence with lines $l = \mathbb{V}(v) \subseteq \mathbb{P}^2 = \mathbb{P}(S_1)$. Thus the line scheme of $Skly_3$ is isomorphic to the dual projective space $(\mathbb{P}^2)^*$.

Moreover if M is a plane module of $Skly_3$, then M must be isomorphic to the left S-module S. Therefore the plane scheme of $Skly_3$ is simply a point.

2.1.5 Noncommutative projective schemes

In this section, we briefly discuss the concept of a noncommutative projective scheme as introduced in [AZ94]. Such structures are prompted by the commutative setting; we employ Serre's theorem (Theorem II.1) illustrate this motivation.

Recall that one can associate to any commutative graded algebra R a pair $(X, \mathcal{O}_X)$, where X=Proj(R) is the projective scheme introduced in §2.1.1 and $\mathcal{O}_X$ is its structure sheaf. The first part of Serre's theorem states that we have an equivalence of categories: coh$(X) \sim R$-qgr. Here the structure sheaf $\mathcal{O}_{\mathrm{Proj}R}$ corresponds to the object $\pi({}_R R)$, where π is the quotient functor $\pi : R\text{-gr} \to R\text{-qgr}$. Hence the commutative projective scheme Proj(R) corresponds the pair (R-qgr, $\pi({}_R R)$).

On the other hand, the second part of Serre's theorem states that the category coh(X) is equivalent to B-qgr, where $B = B(X, \mathcal{L})$ is the section ring where X = Proj(R) and $\mathcal{L}$ an ample invertible sheaf on X. In particular if $\mathcal{L} = \mathcal{O}_X(1)$, then $\mathcal{L}$ corresponds to the image of the shifted module $\pi({}_B B[1])$ under this equivalence.

Moreover the final part of Serre's result pertains to the *polarized variety* $(X, \mathcal{O}_X(1))$; namely that we have an equivalence of categories, R-qgr $\sim B(X, \mathcal{O}_X(1))$-qgr, provided that R is also generated in degree one.

We now present Artin and Zhang's scheme construction pertinent to the (noetherian) noncommutative setting.

Definition II.21. [AZ94, §2] Given a graded left-noetherian (not necessarily commutative) k-algebra A, we say that a **noncommutative projective scheme** is the triple:

$$(A-\text{qgr},\ \pi({}_A A),\ s_A).$$

Here s_A is the autoequivalence on A-qgr induced by the shift functor $M \mapsto M[1]$ in A-gr.

For future reference, we will use this construction in §2.3 and §3.5.

2.2 Gelfand-Kirillov dimension

We introduce in this section the notion of Gelfand-Kirillov dimension, a growth measure of algebras and modules in terms of a generating set. The main reference for this theory is a text of Krause and Lenagen [KL00], yet we refer to various sources for the discussion below.

Notation. Unless stated otherwise, let A be a finitely generated, locally finite, $\mathbb{N}$-graded, connected k-algebra. Take $M \in A$-gr, and put $A_{\leq n} := \bigoplus_{i=0}^{n} A_i$ and $M_{\leq n} := \bigoplus_{i=0}^{n} M_i$.

Definition II.22. [MR01, 8.1.6, 8.1.11] Given finitely generated $M \in A$-gr, define a function $f_M : \mathbb{Z} \to \mathbb{N}$ by $f_M(d) := \sum_{n \leq d} \dim_k M_n$.

1. The **Gelfand-Kirillov dimension** of M, denoted GKdim(M) is defined to be

$$s = \overline{\lim}_d \; \frac{\log f_M(d)}{\log d}.$$

   Equivalently, we have that:

$$\mathrm{GKdim}(M) = \liminf\{\alpha \mid \exists c \geq 0 \text{ so that } \dim_k(M_{\leq n}) \leq cn^{\alpha} \text{ for all } n >> 0\}.$$

2. A module M is s-**critical** if GKdim(M)=s and all of its proper quotients have strictly smaller Gelfand-Kirillov dimension.

Remark II.23. In the thesis, all graded algebras A under consideration have Hilbert series $H_A(t) = (1-t)^{-r}$ for some positive integer r. Hence graded A-modules M have Hilbert series of the form $H_M(t) = q_M(t)(1-t)^{-r}$ for some $q_M(t) \in \mathbb{Z}[t,t^{-1}]$ [ATVdB91, (2.18,2.19)]. Moreover GKdim(M) is the order of the pole at $t = 1$ of $H_M(t)$, to say $H_M(t) = g_M(t)(1-t)^{-\mathrm{GKdim}(M)}$, for some $g_M(t) \in \mathbb{Z}[t,t^{-1}]$.

Another useful invariant of the module $M \in A-\mathrm{gr}$ is given below.

Definition II.24. The **multiplicity** of M is defined to be

$$\mathrm{mult}(M) := g_M(1) = [(1-t)^{\mathrm{GKdim}(M)} H_M(t)]|_{t=1} \in \mathbb{N}.$$

Example II.25. We have that the d-linear modules (Definition II.19) are examples of modules with multiplicity 1 and GK-dimension $d+1$.

Likewise we can define Gelfand-Kirillov dimension for a graded algebra A by GKdim(A) = GKdim$({}_AA)$. In other words, we have:

Definition II.26. Let A be a finitely generated $\mathbb{N}$-graded connected k-algebra. Then the **Gelfand-Kirillov dimension of the algebra** A is

$$\mathrm{GKdim}(A) = \liminf\{\alpha \mid \exists c \geq 0 \text{ so that } \dim_k(A_{\leq n}) \leq cn^{\alpha} \text{ for all } n >> 0\}.$$

For example, the GK-dimension of the commutative polynomial ring $k[x_1, \ldots, x_n]$ is n, whereas the GK-dimension of the free algebra $k\{x_1, \ldots, x_r\}$ is ∞ for $r \geq 2$. We also consider the following properties presented in the next two propositions.

Proposition II.27. *[MR01, 8.1.17, 8.1.18, 8.2.7] [KL00, Chapter 7] [Sta] Let A be a finitely generated, locally finite, $\mathbb{N}$-graded, k-algebra. Take $M \in A$-mod. We have that the following facts hold.*

1. *If A is commutative, then GKdim(A) equals the degree of the Hilbert-Samuel polynomial.*
2. *If A is not commutative, then GKdim(A) need not be an integer.*
3. *GKdim(M) = 0 if and only if M is finite-dimensional.*
4. *If GKdim(M)< 1, then GKdim(M)=0.*
5. *If A is also a connected graded domain, then*

$$\mathit{GKdim}(A) = 1 + \liminf\{\beta \mid \exists c \geq 0 \text{ so that } \dim_k(A_n) \leq cn^{\beta} \text{ for all } n >> 0\}.$$

6. *GKdim($A[t]$) = GKdim(A) + 1*

Proposition II.28. *[KL00, Proposition 3.15] Let I be an ideal of a k-algebra A and assume that I contains a (left or right) regular element of A. Then*

$$\mathit{GKdim}(A/I) \;\leq\; \mathit{GKdim}(A) - 1.$$

Note that GKdim(A)$< \infty$ if and only if A has polynomial bounded growth. We now introduce terms pertaining to the case that GKdim(A)=∞.

Definition II.29. [SZ97] Let A be a graded locally finite k-algebra. We say that A has **sub-exponential growth** if $\overline{\lim}_d\, f_A(d)^{1/d} \leq 1$, and has **exponential growth** if $\overline{\lim}_d\, f_A(d)^{1/d} > 1$.

So if A has polynomial growth, then it has sub-exponential growth. However the converse does not hold.

These notions of growth slightly differ from the standard definitions (say as presented in [KL00]), yet the advantage of the [SZ97] version is that it is used to study the ring-theoretic behavior of A. Namely we have the following result of Stephenson and Zhang, which we use throughout this thesis.

Theorem II.30. *[SZ97, Theorem 0.1] If A is a graded locally finite left noetherian k-algebra, then A has sub-exponential growth.*

2.3 Representation theory via noncommutative projective algebraic geometry

In this section, we provide the background information for the study of the representation theory of certain noncommutative algebras U that are not necessarily graded. We are particularly interested in classifying simple finite-dimensional U-modules, a set denoted by $\mathbf{Simp}_{<\infty}U$. We refer to the set of finite-dimensional U-modules by $\mathbf{Repr}_{<\infty}U$. The idea is to interpret U as analogous to the coordinate ring of an affine open subset of a projective scheme $\mathrm{Proj}(A)$. Classifying $\mathrm{Simp}_{<\infty}A$ is also of interest.

Recall from Definition II.22 the notion of a critical module. We now present an association between irreducible finite-dimensional representations of A and 1-critical A-modules. We also require the following terminology.

Definition II.31. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a connected graded k-algebra. The ideal $A_+ := \bigoplus_{i\geq 1} A_i$ is referred to as the **irrelevant ideal** (or as the **augmentation ideal**). Moreover the module $A/A_+ \cong {}_A k$ is called the **trivial module**.

Lemma II.32. *[SS93, Lemma 4.1] Let A be a connected graded k-algebra generated in degree one that is noetherian and locally finite ($\dim_k A_i < \infty$). Assume that GKdim(A) ≥ 1. Now if M is a nontrivial finite-dimensional simple A-module, then M is a quotient of some 1-critical graded A-module.*

Proof. We repeat the proof presented in [SS93]. Let M be a nontrivial simple finite-dimensional left S-module. Define a new left S-module: $\tilde{M} := M \otimes_k k[t]$, with $s \in S_d$ acting by $s * (m \otimes t^i) = (sm) \otimes t^{i+d}$. Then $\tilde{M}$ is a graded S-module with $\tilde{M}_d = M \otimes (k \cdot t^d)$. By considering the map $\pi : \tilde{M} \to M$ given by $\pi(m \otimes t^d) = m$, we see that M is isomorphic to the factor $\tilde{M}/\tilde{M}(t-1)$. Hence M is the quotient of a graded module $\tilde{M}$, a module which has GK-dimension 1 (Proposition II.27) and multiplicity equal to $\dim_k M$.

Since there is a filtration

$$\tilde{M} = \tilde{M}^0 \supseteq \tilde{M}^1 \supseteq \tilde{M}^2 \supseteq \cdots \supseteq \tilde{M}^k = 0$$

by graded submodules so that $\tilde{M}^i/\tilde{M}^{i+1}$ is critical, we have that M is a quotient of one of these factors. Thus M is the quotient of a 1-critical graded A-module. □

Remark II.33. [SS93, Remarks after Lemma 4.1] Say a nontrivial module M in $\mathrm{Simp}_{<\infty} S$ is the quotient of some 1-critical graded module N. In other words, there is an A-module map $\psi : N \to M$. Then there exists a unique degree 0 A-module map $\tilde{\psi} : N \to \tilde{M}$ such that $\psi = \pi \circ \tilde{\psi}$. This map $\tilde{\psi}$ is given by $\tilde{\psi}(n) = \psi(n) \otimes t^d$ for $n \in N_d$. Now $N[k]$ embeds into $\tilde{M}$ for some $k \in \mathbb{Z}$, and $\mathrm{mult}(N) \leq \mathrm{mult}(\tilde{M}) = \dim_k M$.

The rest of section is dedicated to understanding 1-critical graded A-modules, which is used for the study of $\mathrm{Simp}_{<\infty} A[c^{-1}]_0$ for some central element c of A. Here $A[c^{-1}]_0$ is the algebra U mentioned at the beginning of this section. Moreover we will see that the class of 1-critical graded A-modules is associated to the set of irreducible objects of the category A-qgr.

Notation. We denote by Irred(A-qgr), the subcategory of A-qgr whose objects consist of irreducible objects of A-qgr.

2.3.1 Commutative motivation

To motivate §2.3.2, we illustrate the geometry of 1-critical graded R-modules for R a commutative connected graded ring generated in degree one. Of course if R is commutative, then the finite-dimensional representations of R are 1-dimensional, so this setting is rather different from the noncommutative notions discussed later.

Lemma II.34. *Let R be a commutative graded k-algebra, generated in degree one over $R_0 = k$. We have an equivalence of categories whose objects are respectively:*

(a) 1-critical graded R-modules;

(b) irreducible objects of R-qgr;

(c) R-point modules.

Proof. We remind the reader that the field k is algebraically closed. The equivalence of categories (a) and (c) is discussed in [Smi94, §6], whereas the equivalence between categories (b) and (c) can be understood geometrically via Serre's theorem (Theorem II.1). Namely Irred(R-qgr) $\sim$ Irred(coh(X)), where the latter corresponds to closed points of X=Proj(R). □

Thus by Lemma II.32 when R is commutative, all simple finite-dimensional R-modules arise as simple quotients of point modules.

However when given a noncommutative algebra A, the category A-qgr is generally not equivalent to coh(X) for some projective scheme X. Thus it is useful to consider factors of A for which this scenario occurs. We depict such a scenario for the commutative setting as follows.

Let f be a regular element of R, homogeneous of positive degree. Consider the decomposition of the projective scheme $\mathrm{Proj}(R) =: X$ into a union of a closed subscheme $\mathrm{Proj}(R/(f))$ and its open complement X_f. We aim to investigate this union categorically. First consider $\mathbb{V}(f) = \mathrm{Proj}(R/(f))$, a closed subscheme of X. We have by Theorem II.1 an equivalence of categories:

$$\text{(2.3)} \qquad \mathrm{coh}(\mathbb{V}(f)) \sim \mathrm{Irred}(R/(f) - \mathrm{qgr}) \sim \underset{f\text{-torsion}}{\mathrm{Irred}}(R\text{-qgr}).$$

In other words, the closed subscheme $\mathbb{V}(f)$ of X is thought of categorically as the subcategory $\underset{f\text{-torsion}}{\mathrm{Irred}}(R\text{-qgr})$ of $\mathrm{Irred}(R\text{-qgr})$.

On the other hand, consider the complement X_f of $\mathbb{V}(f)$ in X:

$$X_f = \{p \in X \mid f(p) \neq 0\},$$

or rather the affine scheme $\mathrm{Spec}(R[f^{-1}]_0)$. Instead of considering coherent sheaves on X_f, we use the equivalence of 'complementary' categories below:

$$\text{(2.4)} \qquad \underset{f\text{-torsionfree}}{\mathrm{Irred}}(R\text{-qgr}) \sim \mathrm{Simp}_{<\infty} R[f^{-1}]_0.$$

This equivalence is induced by the localization map: $R\text{-qgr} \longrightarrow R[f^{-1}]_0\text{-mod}$, given by $[M] \mapsto M[f^{-1}]_0$ [ATVdB91, §7]. Hence the open complement X_f of $\mathbb{V}(f)$ in X is viewed categorically as the category $\underset{f\text{-torsionfree}}{\mathrm{Irred}}(R\text{-qgr})$.

We summarize the decompositions discussed above in the following figure. The last diagram, in particular, references the equivalences in (2.3) and (2.4), and Lemmas II.32 and II.34.

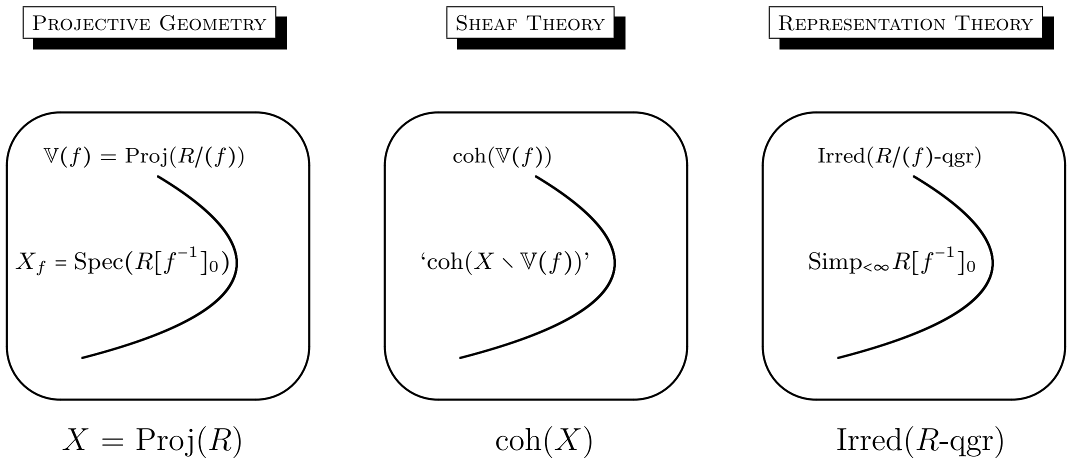

Figure 2.8: (Category equivalences of) the decomposition of R-qgr

2.3.2 Decomposing noncommutative projective schemes (or categories -qgr)

Consider Figure 2.8 with $R = A$, an arbitrary noncommutative connected graded algebra generated in degree 1. Recall that our goal is not only understand the class of finite-dimensional simple A-modules $\mathrm{Simp}_{\infty} A$, but to also classify $\mathrm{Simp}_{<\infty} U$ for some ungraded algebra U prompted by the affine geometry of $\mathrm{Proj}(A)$. From the discussion of §2.1.2, we know that it is undesirable to use $\mathrm{Proj}(A)$, so instead we generalize the relationship between the last two diagrams of Figure 2.8 to the noncommutative setting. In particular, we employ a noncommutative version of Serre's Theorem (Theorem II.14).

Considering the quotient functor $\pi : A\text{-gr} \to A\text{-qgr}$, we now study the representatives $F \in A\text{-gr}$ of irreducible objects in A-qgr.

Proposition II.35. *[Smi94, Proposition 7.1] Every irreducible object in A-qgr has a representative $F \in A$-gr so that:*

1. $F = \bigoplus_{i \geq 0} F_i$;

2. F is generated by F_0;

3. $\dim_k F_i$ is constant for all i, and this constant is equal to mult(F);

4. F has no nonzero finite dimensional submodule.

Hence this prompts the following terminology for the graded A-modules with images in Irred(A-qgr).

Definition II.36. A cyclic graded left A-module M, generated in degree 0, of GK-dimension 1 and multiplicity ϵ is called a **point module** if $\epsilon = 1$, and a **fat point module** if $\epsilon > 1$.

We now use point modules and fat point modules of A to yield results about $\mathrm{Simp}_{<\infty}(A)$. To achieve a decomposition similar to Figure 2.8, first take a central regular element c of A, homogeneous of positive degree. If we are given a module $M \in A$-qgr that is c-torsion, then $M \in A/(c)$-qgr. Otherwise by [ATVdB91, Proposition 7.5], we have that M yields an object in $\mathrm{Repr}_{<\infty} A[c^{-1}]_0$. Hence an irreducible object of A-qgr is either an irreducible object of the category $A/(c)$-qgr or it corresponds to a finite-dimensional simple module over the ungraded algebra $A[c^{-1}]_0$.

A nice scenario occurs when (in addition to the hypotheses on A and c above) the factor $A/(c)$ is isomorphic to a twisted homogeneous coordinate ring $B(\mathcal{P}_A, \mathcal{L}, \sigma)$, where $\mathcal{P}_A$ is the point scheme of A, and $\mathcal{L}$ is a σ-ample invertible sheaf of $\mathcal{P}_A$. Morally we say that the noncommutative projective scheme (A-qgr, $\pi({}_AA)$, s_A) (Definition II.21) decomposes into the "closed subscheme" (B-qgr, $\pi({}_BB)$, s_B), and its open complement corresponding to objects of $\mathrm{Simp}_{<\infty} A[c^{-1}]_0$. Such a notion is the noncommutative analogue of the third diagram of Figure 2.8.

In fact we understand the category, Irred(B-qgr), by Noncommutative Serre's theorem (Theorem II.14); the irreducible objects of B-qgr correspond to B-point modules. Hence by the comments after Definition II.36, finite-dimensional simple modules over $A[c^{-1}]_0$ correspond to fat point modules over A. In other words, classifying fat point modules of A produces results about the representation theory of the (ungraded) algebra $A[c^{-1}]_0$. We will see later that this can be achieved by studying the representation theory of the graded algebras A and $A/(c)$, techniques which are particularly useful in Chapter 4 where we study *deformed Sklyanin algebras* (see Example 3 in §2.3.3).

2.3.3 Examples

We provide examples of the discussion in the previous subsection. Namely we illustrate the decomposition of a noncommutative scheme (A-qgr, $\pi({}_AA)$, s_A) into a closed subscheme and open complement, and to yield results in representation theory.

(1) Let A be the homogenized Weyl algebra,

$$k\{x, y, w\}/(xy - yx - w^2,\ xw - wx,\ yw - wy),$$

and let $c = w$. Let k is an algebraically closed field of characteristic zero. Then $A/(w) = k[x, y]$, a commutative polynomial ring. Moreover $A[w^{-1}]_0$ is isomorphic to the first Weyl algebra $W = k\{x, y\}/(xy - yx - 1)$. Since the Weyl algebra has no irreducible finite-dimensional representations, then all (nontrivial) simple finite-dimensional A-modules arise as simple quotients of 1-critical modules (or point modules in this case) over $k[x, y]$. These can be explicitly computed.

(2) Let A be $Skly_3(a, b, c)$ with $abc \neq 0$ and $(a^3 + b^3 + c^3)^3 \neq (3abc)^3$, and let $c = g$ be the degree 3 central element from Remark II.17. Then $S/(g)$ is isomorphic to a twisted homogeneous coordinate ring $B(E, \mathcal{L}, \sigma)$ for E an elliptic curve. Moreover $S[g^{-1}]_0$ is a simple ring if $|\sigma| = \infty$, and *Azumaya* (Definition II.46) if $|\sigma| < \infty$

[ATVdB91, Proposition 7.3]. Thus one can apply results such as Theorem II.51 to understand simple finite-dimensional modules over $S[g^{-1}]_0$. Further work about the sets $\mathrm{Simp}_{<\infty}S$, $\mathrm{Simp}_{<\infty}B$, and $\mathrm{Simp}_{<\infty}S[g^{-1}]_0$ is presented in §4.3.

(3) In Chapter 4 we will define a large class of *deformed Sklyanin algebra*, S_{def}, for which this discussion of §2.3.2 is particularly useful. The algebras have the following properties. Let $A = D$, a *central extension* of $Skly_3$ (Definition IV.6), and let $c = w$. Then $A/(w) = Skly_3$ and $A[w^{-1}]_0 = S_{def}$ (Definition IV.2).

2.4 Polynomial identity (PI) rings

Polynomial identity (PI) rings have been of interest to algebraists and geometers since the 1920s; refer to [Ami74] for a historical account of these structures. This section will review basic definitions, properties, and examples of PI rings as presented in [MR01, Chapter 13]. Special subclasses of PI rings are also discussed in §§2.4.2-2.4.3.

2.4.1 Basic properties of PI rings

Definition II.37. A **polynomial identity (PI) ring** is a ring A for which there exists a monic multilinear polynomial $f \in \mathbb{Z}\{x_1, \dots, x_n\}$ so that $f(a_1, \dots, a_n) = 0$ for all $a_i \in A$. The minimal degree of such a polynomial is referred as the **minimal degree** of A.

Proposition II.38. *(1) Any ring that is module-finite over its center is PI.*

(2) Any subring or homomorphic image of a PI ring is PI.

Examples II.39. (1) Commutative rings R are PI as the elements satisfy the polynomial identity $f(x, y) = xy - yx$ for all $x, y \in R$.

(2) The ring of $n \times n$ matrices over a commutative ring R, $\mathrm{Mat}_n(R)$, is PI.

(3) The ring $A = k\{x, y\}/(xy + yx)$ is PI as A is a rank 4 module over its center $Z(A) = k[x^2, y^2]$.

In fact if A is PI, then $\mathrm{Mat}_n(A)$ is also PI for all $n \geq 1$.

2.4.2 On central simple algebras (CSAs)

Now we study a nice subset of PI rings: central simple algebras (CSAs); details can be found in [MR01, §13.3].

Definition II.40. A ring is a **central simple algebra (CSA)** if its simple, artinian, and a finite-dimensional module over its center. Hence by Wedderburn's theorem, $A \cong \mathrm{Mat}_n(D)$ for some division ring D, where D is module-finite over its center.

Figure 2.9: The class of PI rings contains the subclass of CSAs

Provided a CSA, we can explicitly measure the ring's noncommutativity. This prompts the notion of the PI degree.

Lemma II.41. *[MR01, Corollary 13.3.5] Let $A = Mat_n(D)$ be a CSA with $Z(D) = C$. Then $\dim_C D = m^2$ and $\dim_C A = (mn)^2 =: p^2$ for some $m, n \in \mathbb{N}$.*

Considering the theorem above, we have that $2p$ is the minimal degree of A. The value p is called the **PI degree of a CSA** A. Moreover, the value p^2 is the rank of A over its center.

The notion of PI degree can be extended to the class of prime PI rings via Posner's theorem. This is discussed in [MR01, 13.6.5] and we restate their remarks here.

Theorem II.42. *(Posner) Let A be a prime PI ring with center Z and with minimal degree $2p$. Let $\mathcal{S} = Z \setminus \{0\}$, let $Q = A\mathcal{S}^{-1}$, and let $F = Z\mathcal{S}^{-1}$ denote the quotient field of Z. Then Q is a CSA with center F and $\dim_F Q = p^2$.*

Considering the notation above, the **PI degree of a prime PI ring** A is equal to $(\dim_F Q)^{1/2}$, where Q is the *Goldie quotient ring* of A.

Examples II.43. (1) Let R be a commutative ring. Then R has minimal degree 2, has PI degree 1, and is rank 1 over its center.
(2) Let R be a commutative ring. Then $\mathrm{Mat}_n(R)$ has minimal degree $2n$, has PI degree n, and is rank n^2 over its center.
(3) The ring $A = k\{x, y\}/(xy + yx)$ has minimal degree 4, has PI degree 2, and is rank 4 over its center.

Now provided that an extra condition holds, a PI ring is in fact a CSA.

Definition II.44. A ring A is said to be **(left) primitive** if A has a faithful simple module M. In other words, M is simple with the condition that $aM = 0$ only if $a = 0$ for any $a \in A$.

Theorem II.45. *[MR01, Theorem 13.3.8] (Kaplansky) If A is a primitive PI ring of minimal degree d, then A is a CSA of rank $(d/2)^2$ over its center.*

2.4.3 On Azumaya algebras

We introduce Azumaya algebras, an extension of the class of central simple algebras (CSAs) which is a well behaved subclass of PI rings. A further discussion of these rings is found in [MR01, §13.7].

Definition II.46. Let A be a ring with center C, and let $E := \mathrm{End} A_C$. Then A is **Azumaya** over C if

(i) A_C is finitely generated and projective, and

(ii) $A \otimes_C A^{op} \longrightarrow E$ is an isomorphism.

Note that the definition above is rarely used in practice. Instead we may want to think of an Azumaya algebra as a CSA over C, where C is a commutative ring that is not necessarily a field. Here are some basic properties.

Proposition II.47. *(1) If A is Azumaya, then A is PI.*

(2) If A is a CSA, then A is Azumaya.

(3) If R is a commutative ring, then $Mat_n(R)$ is Azumaya.

Figure 2.10: The class of PI rings contains Azumaya algebras, which in turn contains CSAs

Again these rings are closely tied to their centers.

Proposition II.48. *[MR01, Proposition 13.7.9] Say A is Azumaya with center $Z(A)$. Then there is a 1-1 correspondence between the set of ideals of A and the set of ideals of $Z(A)$. This is given by the maps: $I \mapsto I \cap Z(A)$ and $HA \leftarrow H$, respectively.*

The above result is useful from the standpoint of noncommutative geometry, as in this setting: $\mathrm{Spec}(A) = \mathrm{Spec}(Z(A))$.

Corollary II.49. *If A is Azumaya, then the following are equivalent:*

(i) A has the a.c.c. on ideals;

(ii) A is right and left Noetherian; and

(iii) $Z(A)$ is Noetherian.

We end with a couple of structure theorems on (the representations of) prime Azumaya algebras.

Theorem II.50. *[MR01, Theorem 13.7.14] (Artin-Procesi) Let A be a prime ring. Then the following are equivalent*

(i) A is Azumaya of rank n^2;

(ii) A is a prime PI ring so that for all primes $\mathfrak{p}$: PIdeg($A/\mathfrak{p}$) = PIdeg(A), with PIdeg(A) = n.

Pertaining to Chapter 4 in particular, it is worth pointing out that prime Azumaya algebras are nice in terms of representation theory due to the following result of Artin.

Theorem II.51. *[Art69] Let A be a prime Azumaya algebra. Then the dimension of a simple finite-dimensional A-module is equal to the PI degree of A.*

We can apply this discussion to arbitrary PI algebras provided we restrict our attention to the *Azumaya locus* as we define below.

Proposition II.52. *[BG97, Proposition 3.1] Let A be a prime noetherian ring that is module finite over its center Z. Assume that Z is an affine algebra over an algebraically closed field k. Then the following statements hold.*
(a) The maximum k-vector space dimension of simple finite-dimensional A-modules is the PI degree of A.
(b) Let M be a simple A-module, let P denote $ann_A M$, and let $\mathfrak{m} = P \cap Z$. Then $\dim_k M$ = *PIdeg A if and only if $A_{\mathfrak{m}}$ is an Azumaya over $Z_{\mathfrak{m}}$. Here $A_{\mathfrak{m}}$ is the localization $A \otimes_Z Z_{\mathfrak{m}}$.*

The condition in part (b) prompts the following definition.

Definition II.53. Assume that the hypotheses of Proposition II.52 hold. Consider the sets:

$$\begin{aligned}\mathcal{A}_A &= \{\mathfrak{m} \in \text{maxspec}Z \mid A_{\mathfrak{m}} \text{ is Azumaya over } Z_{\mathfrak{m}}\},\\ \mathcal{S}_A &= \{\mathfrak{m} \in \text{maxspec}Z \mid \text{ gldim}(Z_{\mathfrak{m}}) = \infty\}.\end{aligned}$$

These are called the **Azumaya locus** and **singular locus** of A respectively. The ideals in the Azumaya locus are referred to as **Azumaya points**. Moreover the complement of $\mathcal{S}_A$ in maxspec$Z$ is referred to as the **smooth locus**.

Thus when A is Azumaya, we have that $\mathcal{A}_A = \text{maxspec}Z$ and we recover the result of Theorem II.51. More generally if the hypotheses of Proposition II.52 hold, then we get that:

Proposition II.54. *[BG02, §III.1] $\mathcal{A}_A$ is an open, Zariski dense subset of maxspec$Z$.*

Moreover we have that $\mathcal{A}_A$ and $\mathcal{S}_A$ are related in the following manner.

Lemma II.55. *[BG97, Lemma 3.3] Assume the hypotheses of Proposition II.52. If gldim$(A) < \infty$, then $\mathcal{A}_A \subseteq \text{maxspec}Z \smallsetminus \mathcal{S}_A$.*

Equality can actually be achieved, to say the Azumaya and smooth loci of A can coincide, provided additional conditions on A hold [BG97, Theorem 3.8].

Therefore we see that the representation theory of noncommutative PI k-algebras can be analyzed geometrically.

2.5 Bergman's Diamond Lemma

This section is dedicated to results of Bergman, which construct a k-vector space basis for a given k-algebra A. The details for the following work is found in [Ber78], and the presentation of this material is borrowed from [Sta].

Notation. Given a k-algebra $A = T(V)/I$ where V is a finite-dimensional k-vector space, pick an ordered basis $x_1 < x_2 < \cdots < x_n$ for V. We form a basis of $T(V)$ by considering the set of all monomials in $\{x_i\}$ under the lexicographical ordering on $< V >$. Let I be generated as an ideal by $\{s_i\}$ where (after nonzero scalar multiplication) $s_i = w_i - f_i$, so that w_i is a monic monomial $> f_i$.

We introduce a way of simplifying elements of A by using the following maps.

Definition II.56. Given monomials $\mathtt{m}, \mathtt{n} \in < V >$, we have maps:

$$\sigma :< V > \longrightarrow T(V), \quad \text{given by} \quad \mathtt{m} w_i \mathtt{n} \mapsto \mathtt{m} f_i \mathtt{n}.$$

This extends by linearity to: $T(V) \longrightarrow T(V)$; products of these maps are called **reductions**.

Definition-Lemma II.57. *For all $t \in T(V)$, we have that t is* **reduction finite** *in the sense that for reductions σ_i, we have that: $\sigma_s \cdots \sigma_1(t) = \sigma_{s+1}\sigma_s \cdots \sigma_1(t)$, $\forall s >> 0$.*

Now considering reductions, we define the following terms of $T(V)$.

Definition II.58. (a) We say that $t \in T(V)$ is **irreducible** if $\sigma(t) = t$ for all reductions σ. Let $T(V)_{irred}$ denote the set of irreducible elements of $T(V)$. This is clearly spanned by $< V >_{irred}$.
(b) We say $t \in T(V)$ is **reduction unique** if for all sequences of reductions from t to an irreducible element, then we get the same element. In this case, write this unique element as $\sigma_\infty(t)$.

Example II.59. Consider the k-algebra $A = k\{x, y\}/(yx - xy - x^2)$.
(a) Take $y > x$ so $w = yx$ and $f = xy + x^2$. Then we see that the set of irreducible monomials of A are of the form $\{x^i y^j\}_{i,j \in \mathbb{N}}$.
(b) Take $x > y$ so $w = x^2$ and $f = xy - yx$. The element x^3 is not reduction unique due to the following computations:

$$\begin{aligned} x^3 &= x^2 \cdot x = (xy - yx)x &&= xyx - y(x^2) \\ &= xyx - y(xy - yx) &&= xyx - yxy + y^2x \end{aligned}$$

and

$$\begin{aligned} x^3 &= x \cdot x^2 = x(xy - yx) &&= (x^2)y - xyx \\ &= (xy - yx)y - xyx &&= xy^2 - yxy - xyx. \end{aligned}$$

Therefore $2xyx + y^2x - xy^2 = 0$ is a *hidden relation*. Continue by including this hidden relation in the ideal of relations I of A and repeat reductions for the other monomials that are potentially not reduction unique. For a given degree, this process eventually terminates and we get a basis of irreducible monomials of the form

$$\{x^\epsilon \cdot y^{i_1} \cdot x \cdot y^{i_2} \cdot x \cdot y^{i_3} \cdots\}$$

for $i_j \in \mathbb{N}$, $\epsilon = 0, 1$.

We now examine conditions where reduction uniqueness fails.

Definition II.60. (a) Say that there exist monomials $\mathtt{m}, \mathtt{n}, \mathtt{p} \in V$ so that $\mathtt{mn} = w_1$ and $\mathtt{np} = w_2$ for some relations $s_1 = w_1 - f_1$ and $s_2 = w_2 - f_2$. This is called an **overlap ambiguity**. It is *resolvable* (we are able to find σ_∞) provided that $f_1\mathtt{p}$ and $\mathtt{m}f_2$ are reduction unique with $\sigma_\infty(f_1\mathtt{p}) = \sigma_\infty(\mathtt{m}f_2)$.
(b) Say that there exist monomials $\mathtt{m}, \mathtt{n}, \mathtt{p} \in V$ so that $\mathtt{mnp} = w_1$ and $\mathtt{n} = w_2$ for some relations $s_1 = w_1 - f_1$ and $s_2 = w_2 - f_2$. This is called an **inclusion ambiguity**. It is *resolvable* provided that $\sigma_\infty(f_1) = \sigma_\infty(\mathtt{m}f_2\mathtt{p})$.

Overlap ambiguity

$$\begin{tikzcd}
& \mathtt{mnp} \arrow[dl, squiggly, "="'] \arrow[dr, squiggly, "="] & \\
w_1\mathtt{p} \arrow[d, squiggly] & & \mathtt{m}w_2 \arrow[d, squiggly] \\
f_1\mathtt{p} \arrow[dr, squiggly] & & \mathtt{m}f_2 \arrow[dl, squiggly] \\
& \sigma_\infty(f_1\mathtt{p}) = \sigma_\infty(\mathtt{m}f_2) &
\end{tikzcd}$$

Inclusion ambiguity

$$\begin{tikzcd}
& \mathtt{mnp} \arrow[dl, squiggly, "="'] \arrow[dr, squiggly, "="] & \\
w_1 \arrow[d, squiggly] & & \mathtt{m}w_2\mathtt{p} \arrow[d, squiggly] \\
f_1 \arrow[dr, squiggly] & & \mathtt{m}f_2\mathtt{p} \arrow[dl, squiggly] \\
& \sigma_\infty(f_1) = \sigma_\infty(\mathtt{m}f_2\mathtt{p}) &
\end{tikzcd}$$

Figure 2.11: Resolution of ambiguities

The following theorem provides us with the precise conditions for which we can write down an irreducible basis of monomials for the algebra $A = T(V)/I$. If A is graded, then the theorem yields an algorithm for computing such a basis.

Theorem II.61. *[Ber78, Theorem 1.2] (Bergman's Diamond Lemma)*

The following are equivalent:

(1) $A = T(V)/I$ *has a k-vector space basis of monomials* $< V >_{irred}$*,*

(2) All elements of $T(V)$ *are reduction unique,*

(3) All ambiguities of A *are resolvable.*

Below are two examples of Diamond Lemma computations that are required in Chapter 3.

Example II.62. Consider the algebra $S = S(1,b,c)$ in Lemma III.8. Namely we have that $b^3 = c^3 = 1$ and

$$S = k\{x,y,z\}/\left(yz + bzy + cx^2,\ zx + bxz + cy^2,\ xy + byx + cz^2\right).$$

Take the ordering $x < y < z$. We first show that all ambiguities of S are resolvable. Secondly we compute a basis of irreducible monomials of S. Observe that $b^{-1} = b^2$ and $c^{-1} = c^2$. Now rewrite the relations of S as follows:

$$\begin{aligned} z^2 &= -bc^2yx - c^2xy, \\ zy &= -b^2yz - b^2cx^2, \\ zx &= -bxz - cy^2. \end{aligned}$$

There are three (overlap) ambiguities that we must resolve:

$$\begin{aligned} z^3 &= z^2 \cdot z = z \cdot z^2, \\ z^2y &= z^2 \cdot y = z \cdot zy, \\ z^2x &= z^2 \cdot x = z \cdot zx. \end{aligned}$$

Here are the computations for the first ambiguity:

$$z^2 \cdot z \quad = (-bc^2yx - c^2xy) \cdot z \ = -bc^2yxz - c^2xyz$$

and

$$\begin{aligned} z \cdot z^2 &= z \cdot (-bc^2yx - c^2xy) \ = -bc^2(zy)x - c^2(zx)y \\ &= c^2y(zx) + x^3 + bc^2x(zy) + y^3 \\ &= -bc^2yxz - y^3 + x^3 - c^2xyz - x^3 + y^3 \\ &= -bc^2yxz - c^2xyz. \end{aligned}$$

Hence the ambiguity z^3 is resolvable.

Likewise $z^2 \cdot y = -bc^2yxy - c^2xy^2$ and

$$\begin{aligned} z \cdot zy &= z \cdot (-b^2yz - b^2cx^2) \; = -b^2(zy)z - b^2c(zx)x \\ &= by(z^2) + bcx^2z + cx(zx) + b^2c^2y^2x \\ &= -b^2c^2y^2x - bc^2yxy + bcx^2z - bcx^2z - c^2xy^2 + b^2c^2y^2x \\ &= -bc^2yxy - c^2xy^2. \end{aligned}$$

Thus z^2y is resolvable. Similarly, we have that $z^2 \cdot x = z \cdot zx = -bc^2yx^2 - c^2xyx$.

Now by Theorem II.61, we can then conclude that S has a k-vector space of monomials $< V >_{irred}$. Moreover we have that $< V >_{irred}$ equals the set of monomials in $k\{x,y\} \oplus k\{x,y\} \cdot z$.

Example II.63. Given the dual $S^!$ of S in Example II.62, we show that all ambiguities are resolvable. Here $S^! = k\{u,v,w\}/(R)$ where R is the set of relations:

$$\{w^2 = cuv,\; wv = b^2vw,\; wu = buw,\; vw = c^2u^2,\; v^2 = bcuw,\; vu = b^2uv\}.$$

Take the ordering $u < v < w$, and recall that $b^3 = c^3 = 1$. There are twelve (overlap) ambiguities, which by the following calculations are resolvable:

$$\begin{array}{lllllll} w^3 &=& w^2 \cdot w &=& w \cdot w^2 &=& cu^3, \\ w^2v &=& w^2 \cdot v &=& w \cdot wv &=& bc^2u^2w, \\ w^2u &=& w^2 \cdot u &=& w \cdot wu &=& b^2cu^2v, \\ wvw &=& wv \cdot w &=& w \cdot vw &=& b^2c^2u^2w, \\ wv^2 &=& wv \cdot v &=& w \cdot v^2 &=& b^2c^2u^2v, \\ wvu &=& wv \cdot u &=& w \cdot vu &=& b^2c^2u^3, \end{array} \qquad \begin{array}{lllllll} vw^2 &=& vw \cdot w &=& v \cdot w^2 &=& c^2u^2w, \\ vwv &=& vw \cdot v &=& v \cdot wv &=& c^2u^2v, \\ vwu &=& vw \cdot u &=& v \cdot wu &=& c^2u^3, \\ v^2w &=& v^2 \cdot w &=& v \cdot vw &=& bc^2u^2v, \\ v^3 &=& v^2 \cdot v &=& v \cdot v^2 &=& u^3, \\ v^2u &=& v^2 \cdot u &=& v \cdot vu &=& b^2cu^2w. \end{array}$$

By Theorem II.61, $S^!$ has $< V >_{irred}$ as a k-vector space basis of monomials, and this is precisely the set of the monomials in $k[u] \oplus k[u] \cdot v \oplus k[u] \cdot w$.

CHAPTER III

Degenerate Sklyanin algebras and generalized twisted homogeneous coordinate rings

A vital development in the field of noncommutative projective algebraic geometry is the investigation of connected graded noncommutative rings A with use of geometric data. In particular Artin-Tate-van den Bergh constructed twisted homogeneous coordinate rings B (Definition II.9), whose role is analogous to section rings in classical algebraic geometry. Often B is a well behaved ring and the algebra A will share its properties.

For example consider $A = Skly_3$, the three-dimensional Sklyanin algebra for which we recall the definition for the reader's convenience.

Definition III.1. The **three-dimensional Sklyanin algebras**, denoted by $S(a,b,c)$ or $Skly_3$, are generated by three noncommuting variables x, y, z, subject to three relations:

$$\begin{aligned} &ayz + bzy + cx^2 = 0\\ &azx + bxz + cy^2 = 0 \qquad (3.1)\\ &axy + byx + cz^2 = 0 \end{aligned}$$

for $[a:b:c] \in \mathbb{P}^2_k \smallsetminus \mathfrak{D}$ where

$$\mathfrak{D} = \{[0:0:1],[0:1:0],[1:0:0]\} \ \cup\ \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

The geometric data associated to $Skly_3$ is E = a cubic curve or $\mathbb{P}^2$ (Proposition II.15), from which we can build a twisted homogeneous coordinate ring $B(E)$. Moreover there exists a ring surjection from $Skly_3$ onto $B(E)$ with kernel generated by a regular, central element, homogeneous of degree 3 (Theorem II.16) in the case that E is smooth. Furthermore in this case, the behavior of $Skly_3$ reflects that of B ring-theoretically (Corollary II.18).

During the last few years, a number of examples of (even noetherian) algebras have appeared for which the techniques of Artin-Tate-van den Bergh are inapplicable. This is because the point modules for these algebras cannot by parameterized by a projective scheme of finite type (see for example [KRS05]). Consequently one cannot form a corresponding twisted homogeneous coordinate ring. In this chapter, we explore a recipe suggested in [ATVdB90, §3] for building a generalized twisted homogeneous coordinate ring for *any* connected graded ring. Note that such results appear in [Wal09]. In particular, we provide a geometric approach to examine degenerations of Sklyanin algebras.

Definition III.2. The rings $S(a,b,c)$ from Definition III.1 with $[a:b:c] \in \mathfrak{D}$ are called **degenerate Sklyanin algebras**, denoted by S_{deg}.

We will see that the geometry of S_{deg} involves a degenerate cubic curve. On the other hand we establish algebraic properties of S_{deg} such as growth, dimensions, and other ring-theoretic characteristics in §3.1. These are listed in Theorem I.6.

Motivated by the methods of [ATVdB90], we make use of point modules over S_{deg} to construct the generalized coordinate ring. To do so, we first generate geometric data for S_{deg} in §3.2. We remind the reader of the definitions of (truncated) point modules in Definitions II.3 and II.2.

Theorem III.3. *(Proposition III.26) For $d \geq 2$, the truncated point schemes $V_d \subset (\mathbb{P}^2)^{\times d}$ corresponding to S_{deg} are isomorphic to either:*

(i) a union of three copies of $(\mathbb{P}^1)^{\times \frac{d-1}{2}}$ and three copies of $(\mathbb{P}^1)^{\times \frac{d+1}{2}}$, if d is odd, or;

(ii) a union of six copies of $(\mathbb{P}^1)^{\times \frac{d}{2}}$, if d is even.

Observe that the point scheme of a degenerate Sklyanin algebra does not stabilize to produce a projective scheme of finite type. Hence we cannot mimic the approach of [ATVdB90] to construct a twisted homogeneous coordinate ring associated to S_{deg}. Instead we form a graded ring dependent on the truncated point schemes V_d of S_{deg}. Such a ring is called a *point parameter ring* $P(S_{deg})$ (Definition III.29), which have not appeared in the literature since their introduction in [ATVdB91]. The next results of §5.3 describe the behavior of the first non-trivial examples of these rings.

Theorem III.4. *(Proposition III.31, Theorem III.36, Corollaries III.40 and III.41) The point parameter ring $P = P(S_{deg})$ is generated in degree one. Since $(S_{deg})_1 \cong (P(S_{deg}))_1$, we have that P is a factor of the corresponding S_{deg}. It has Hilbert series $H_P(t) = (1+t^2)(1+2t)[(1-2t^2)(1-t)]^{-1}$. Moreover P has exponential growth and is neither right noetherian, Koszul, nor a domain.*

Thus Theorem III.4 yields a result surprisingly similar to Theorem II.16 (pertaining to $Skly_3 \twoheadrightarrow B(E)$), despite the fact that the rings $Skly_3$ and S_{deg} are entirely different as we will see in §3.1.

Moreover with the results of §3.1, we see that the behavior of $P(S_{deg})$ resembles that of S_{deg} and it is natural to ask if other noncommutative algebras can be analyzed in a similar fashion. Some further directions are discussed in §3.4; those particularly pertaining to the field of coherent noncommutative algebraic geometry are mentioned in §3.5.

3.1 Structure of degenerate Sklyanin algebras

In this section, we establish algebraic properties of degenerate Sklyanin algebras. We begin by considering the degenerate Sklyanin algebras $S(a,b,c)_{deg}$ with $a^3 = b^3 = c^3 = 1$ (Definition III.2) and the following definitions from [GW04].

Definition III.5. Let α be an endomorphism of a ring R. An α**-derivation** on R is any additive map $\delta : R \to R$ so that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. The set of α-derivations of R is denoted α-Der(R).

We write $S = R[z;\alpha,\delta]$ provided S is isomorphic to the polynomial ring $R[z]$ as a left R-module but with multiplication given by $zr = \alpha(r)z + \delta(r)$ for all $r \in R$. Such a ring S is called an *Ore extension of* R. More precisely:

Definition III.6. Let R be a ring, α a ring endomorphism of R, and δ an α-derivation on R. We shall write $S = R[z;\alpha,\delta]$ provided

(a) S is a ring, containing R as a subring;

(b) z is an element of S;

(c) S is a free left R-module with basis $\{1, z, z^2, \dots\}$;

(d) $zr = \alpha(r)z + \delta(r)$ for all $r \in R$.

Such a ring S is called an **Ore extension of** R**.**

By generalizing the work of [BS] we see that most degenerate Sklyanin algebras are factors of Ore extensions of the free algebra on two variables. Note that if $a \neq 1$, then $S(a,b,c)$ is isomorphic to $S\left(1, \frac{b}{a}, \frac{c}{a}\right)$. Hence whenever we consider such an S, we assume that $a = 1$.

Proposition III.7. *Let $a, b, c \in k$ so that $a = 1$ and $b^3 = c^3 = 1$, thus $[a : b : c] \in \mathfrak{D}$. Then we get the ring isomorphism*

$$S(1, b, c) \cong \frac{k\{x, y\}[z; \alpha, \delta]}{(\Omega)} \tag{3.2}$$

where

(a) *$\alpha \in End(k\{x, y\})$ is defined by $\alpha(x) = -bx$, $\alpha(y) = -b^2 y$,*

(b) *$\delta \in \alpha$-$Der(k\{x, y\})$ is given by $\delta(x) = -cy^2$, $\delta(y) = -b^2 cx^2$,*

(c) *$\Omega = xy + byx + cz^2$, which is a normal element of $k\{x, y\}[z, \alpha, \delta]$.*

Proof. By direct computation α and δ are indeed an endomorphism and α-derivation of $k\{x, y\}$ respectively. Moreover $x \cdot \Omega = \Omega \cdot bx$, $y \cdot \Omega = \Omega \cdot by$, $z \cdot \Omega = \Omega \cdot z$ so Ω is a normal element of the Ore extension. Thus both rings of (3.2) have the same generators and relations. □

Lemma III.8. *The Hilbert series of S_{deg} is $H_{S_{deg}}(t) = (1 + t)(1 - 2t)^{-1}$.*

Proof. One can find a basis of irreducible monomials via Bergman's Diamond lemma (Theorem II.61) to imply that $\dim_k S_d = 2^{d-1}3$ for $d \geq 1$. For $S(1, b, c)$, we have shown in Example II.62 that S is free with a basis $\{1, z\}$ as a left or right module over $k\{x, y\}$.

Now consider the monomial algebra $S(1, 0, 0) = k\{x, y, z\}/(yz, zx, xy)$. An irreducible word of S is precisely constructed by following the variable x with either x or z, by following the variable y with either x or y, and by following the variable z with either y or z. Here again $\dim_k S_d = 2^{d-1}3$ for $d \geq 1$. Similar results hold for the other two degenerate Sklyanin algebras that are monomial algebras.

Therefore in each case the algebra S_{deg} has Hilbert series,

$$H_{S_{deg}}(t) \;=\; 1 + \sum_{d \geq 1} 2^{d-1} 3 \cdot t^d \;=\; 1 + 3t \sum_{d \geq 0} (2t)^d \;=\; 1 + \frac{3t}{1 - 2t} \;=\; \frac{1 + t}{1 - 2t}.$$

□

Therefore by Proposition III.7 (for $a^3 = b^3 = c^3 = 1$) or Lemma III.8 (in general) we have the following consequence. Recall the notions of growth introduced in Chapter 2, section 2.

Corollary III.9. *The degenerate Sklyanin algebras have exponential growth, infinite GK dimension, and are not left noetherian. Furthermore S_{deg} is not a domain.*

Proof. For all values $[a:b:c] \in \mathfrak{D}$, we have by Lemma III.8 that

$$r_d := \sum_{i \le d} \dim_k S_i \;=\; \sum_{i \le d} 2^{i-1}3 \;=\; 3(2^{d+1}-1).$$

Since $\overline{\lim}_d\, r_d^{1/d} = 2 > 1$, the algebras S_{deg} have exponential growth. Thus GKdim(S_{deg})= ∞ and so by Theorem II.30, S_{deg} is also not noetherian.

Now if $[a:b:c] \in \{[1:0:0],\ [0:1:0],\ [0:0:1]\}$, then the monomial algebra $S(a,b,c)$ is obviously not a domain. On the other hand if $[a:b:c]$ satisfies $a^3 = b^3 = c^3 = 1$, then assume without loss of generality that $a = 1$. As a result we have

$$f_1 + bf_2 + cf_3 \;=\; (x + by + bc^2 z)(cx + cy + b^2 z),$$

where $f_1 = yz + bzy + cx^2$, $f_2 = zx + bxz + cy^2$, and $f_3 = xy + byx + cz^2$ are the relations of $S(1,b,c)$. Hence S is not a domain. $\square$

Now we verify homological properties of degenerate Sklyanin algebras.

Definition III.10. Let A be a connected graded algebra which is locally finite ($\dim_k A_i < \infty$). We say that A is **Koszul** if its trivial module $A/\bigoplus_{i \ge 1} A_i \cong {}_A k$ admits a minimal resolution $P_\bullet^{\min}$ that is linear. In other words, the entries of the matrices that determine $P_\bullet^{\min}$ belong to A_1. Refer to [Kra, §4] for a discussion of basic properties and examples of such algebras.

Proposition III.11. *The degenerate Sklyanin algebras are Koszul with infinite global dimension.*

Proof. For $S = S(a,b,c)$ with $a^3 = b^3 = c^3 = 1$, assume that $a = 1$ and consider the description of S in Proposition III.7. Since $k\{x,y\}$ is Koszul, the Ore extension $k\{x,y\}[z,\alpha,\delta]$ is also Koszul [CS08, Definition 1.1, Theorem 10.2]. By Proposition III.7, the element Ω is normal and regular in $k\{x,y\}[z;\alpha,\delta]$. Hence the factor S is Koszul by [ST01, Theorem 1.2].

To conclude $\text{gl.dim}(S) = \infty$, note that the Koszul dual of S is $S^! = k\{u,v,w\}/(R)$ where R is the set of relations:

$$\{w^2 = cuv,\ wv = b^2vw,\ wu = buw,\ vw = c^2u^2,\ v^2 = bcuw,\ vu = b^2uv\}.$$

Taking the ordering $u < v < w$, we have shown in Example II.63 that $S^!$ has a basis of irreducible monomials $\{u^i, u^jv, u^kw\}_{i,j,k\in\mathbb{N}}$. Hence $S^!$ is not a finite dimensional k-vector space and by [Kra, Corollary 5], S has infinite global dimension.

For $S = S(a,b,c)$ with $[a:b:c] \in \{[1:0:0],[0:1:0],[0:0:1]\}$, note that S is Koszul as its ideal of relations is generated by quadratic monomials [PP05, Corollary 4.3]. Denote these monomials m_1, m_2, m_3. The Koszul dual of S in this case is $S^! = k\{u,v,w\}/(R)$ where (R) is the ideal of relations generated by the six monomials not equal to m_i (in variables u,v,w). Since $S^!$ is again a monomial algebra, it contains no hidden relations and has a nice basis of irreducible monomials. In particular, $S^!$ contains $\bigoplus_{i\geq 0} kp_i$ where p_i is the length i word:

$$p_i = \begin{cases} \underbrace{uvwuvwu\dots}_{i}, & \text{if } [a:b:c] = [1:0:0] \\ \underbrace{uwvuwvu\dots}_{i}, & \text{if } [a:b:c] = [0:1:0] \\ u^i, & \text{if } [a:b:c] = [0:0:1]. \end{cases}$$

Therefore $S^!$ is not a finite dimensional k-vector space. By [Kra, Corollary 5] these degenerate Sklyanin algebras have infinite global dimension. □

We compute the center of the degenerate Sklyanin algebras.

Proposition III.12. *The center of S_{deg} is equal to k.*

Proof. For $a^3 = b^3 = c^3 = 1$, we assume that $a = 1$ as $S(a,b,c) \cong S\left(1, \frac{b}{a}, \frac{c}{a}\right)$. By Example II.62, an element f of $S(1,b,c)$ can be written uniquely as $f = t_1 + t_2 z$ with $t_i \in k\{x,y\}$. Suppose that f is central. Then $fx = xf$ implies that $t_1 x + t_2 zx = xt_1 + xt_2 z$. Since $zx + bxz + cy^2 = 0$ is a relation of $S(1,b,c)$, we have that

$$t_1 x + t_2(-bxz - cy^2) = xt_1 + xt_2 z.$$

Considering the coefficients of z, we get that

$$\begin{aligned} t_1 x - ct_2 y^2 &= xt_1 \\ -bt_2 x &= xt_2. \end{aligned}$$

Therefore $t_2 = 0$ and $t_1 \in k[x]$. Likewise by using $fy = yf$, we conclude that $t_2 = 0$ and $t_1 \in k[y]$. Hence $f \in k[x] \cap k[y] = k$.

Now take the monomial algebra $S(1,0,0) = k\{x,y,z\}/(yz, zx, xy)$ whose irreducible basis of monomials is described in the proof of Lemma III.8. Take a central element f of $S(1,0,0)$. Now $fx = xf$ implies that either $f \in k[x]$ or $f = \sum \gamma_i m_i$ where $\gamma_i \in k$ and m_i is a monomial of the form $y \cdots z$. Likewise $fy = yf$ implies that $f \in k[y]$ or $f = \sum \gamma_i n_i$ where n_i is a monomial of the form $z \cdots x$. Finally $fz = zf$ yields $f \in k[z]$ or $f = \sum \gamma_i p_i$ where p_i is a monomial of the form $x \cdots y$. The simultaneous occurrence of the three outcomes above implies that $f \in k$. Thus $Z(S(1,0,0)) = k$. With similar reasoning, we conclude that $Z(S(0,1,0)) = Z(S(0,0,1)) = k$. □

3.2 Truncated point schemes of S_{deg}

The goal of this section is to prove Theorem III.3, i.e. to construct the family of truncated point schemes $\{V_d \subseteq (\mathbb{P}^2)^{\times d}\}$ associated to the degenerate three-dimensional Sklyanin algebras S_{deg} (see Definition III.2). These schemes will be used in §3.3 for the construction of a generalized twisted homogeneous coordinate ring, namely the point parameter ring (Definition III.29).

Recall the definitions of truncated point modules of length d and point modules given respectively in Definitions II.3 and II.2. We will see in Lemma III.15 that the family $\{V_d\}$ plays a role in the study of point modules over S_{deg}. First we note the result below which is a special case of [ATVdB90, Corollary 3.13]. This result describes the relationship the isomorphism classes of truncated point modules and of point modules over $S(a,b,c)$ for *any* $[a:b:c] \in \mathbb{P}^2$.

Lemma III.13. *Let $S = S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$. Denote by Γ the set of isomorphism classes of point modules over S and Γ_d the set of isomorphism classes of truncated point modules $M = \bigoplus_{i=0}^{d} M_i$ of length $d+1$. With respect to the truncation function $\rho_d : \Gamma_d \to \Gamma_{d-1}$ given by $M \mapsto M/M_d$, we have that as a set Γ is the projective limit of $\{\Gamma_d\}$.* □

Now we proceed to construct schemes V_d that will parameterize length d truncated point modules. We remind the reader that length d truncated point schemes, V_d, were defined in Definition II.4. We now present the definition explicitly for the algebras $S(a,b,c)$.

Definition III.14. [ATVdB90, §3] The **truncated point scheme of length d, $V_d \subseteq (\mathbb{P}^2)^{\times d}$ for** $S(a,b,c)$, is the scheme defined by the multilinearizations of relations of $S(a,b,c)$ from Definition III.1. More precisely $V_d = \mathbb{V}(f_i, g_i, h_i)_{0 \le i \le d-2}$ where

$$
\begin{aligned}
f_i &= a y_{i+1} z_i + b z_{i+1} y_i + c x_{i+1} x_i \\
g_i &= a z_{i+1} x_i + b x_{i+1} z_i + c y_{i+1} y_i \\
h_i &= a x_{i+1} y_i + b y_{i+1} x_i + c z_{i+1} z_i.
\end{aligned} \tag{3.3}
$$

For example, $V_1 = \mathbb{V}(0) \subseteq \mathbb{P}^2$ so $V_1 = \mathbb{P}^2$. Similarly, $V_2 = \mathbb{V}(f_0, g_0, h_0) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$.

Lemma III.15. *The set Γ_d is parameterized by the scheme V_d.*

Proof. This is a special case of [ATVdB90, Proposition 3.9]. □

In short, to understand point modules over $S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$, Lemmas III.13 and III.15 imply that we can now restrict our attention to truncated point schemes V_d. We point out another useful result pertaining to V_d associated to $S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$.

Lemma III.16. *The truncated point scheme V_d lies in d copies of $E \subseteq \mathbb{P}^2$ where E is the cubic curve $E : (a^3 + b^3 + c^3)xyz - (abc)(x^3 + y^3 + z^3) = 0$.*

Proof. Let p_i denote the point $[x_i : y_i : z_i] \in \mathbb{P}^2$ and

$$
\mathbb{M}_{abc,i} := \begin{pmatrix} c x_i & a z_i & b y_i \\ b z_i & c y_i & a x_i \\ a y_i & b x_i & c z_i \end{pmatrix} \in \mathrm{Mat}_3(k x_i \oplus k y_i \oplus k z_i). \tag{3.4}
$$

The point $p = (p_0, p_1, \ldots, p_{d-1}) \in V_d \subseteq (\mathbb{P}^2)^{\times d}$ must satisfy $f_i = g_i = h_i = 0$ for $0 \le i \le d-2$ by definition of V_d. In other words, one is given $\mathbb{M}_{abc,j} \cdot (x_{j+1} \ y_{j+1} \ z_{j+1})^T = 0$ or equivalently $(x_j \ y_j \ z_j) \cdot \mathbb{M}_{abc,j+1} = 0$ for $0 \le j \le d-2$. Therefore for $0 \le j \le d-1$, $\det(\mathbb{M}_{abc,j}) = 0$. This implies that $p_j \in E$ for each j. Thus $p \in E^{\times d}$. □

3.2.1 On the truncated point schemes of some S_{deg}

We will show that to study the truncated point schemes V_d of degenerate Sklyanin algebras, it suffices to understand the schemes of specific four degenerate Sklyanin algebras. We use the following notion of a Zhang twist.

Definition III.17. Given a $\mathbb{Z}$-graded k-algebra $S = \bigoplus_{n\in\mathbb{Z}} S_n$ with graded automorphism σ of degree 0 on S, we form a **Zhang twist** S^σ of S by preserving the same additive structure on S, and defining multiplication $*$ as follows: $a * b = ab^{\sigma^n}$ for $a \in S_n$.

The following is a special case of [Zha96, Theorem 1.2].

Theorem III.18. *If S is connected graded and generated in degree one, then so is the algebra S^σ. We also have an equivalence of categories: S-Gr $\sim S^\sigma$-Gr.* □

Realize $\mathfrak{D}$ from Definition III.1 as the union of three point sets Z_i:

$$\begin{aligned}
Z_1 &:= \{[1:1:1],\quad [1:\zeta:\zeta^2],\quad [1:\zeta^2:\zeta]\},\\
Z_2 &:= \{[1:1:\zeta],\quad [1:\zeta:1],\quad [1:\zeta^2:\zeta^2]\},\\
Z_3 &:= \{[1:\zeta:\zeta],\quad [1:1:\zeta^2],\quad [1:\zeta^2:1]\},\\
Z_0 &:= \{[1:0:0],\quad [0:1:0],\quad [0:0:1]\}.
\end{aligned} \tag{3.5}$$

where $\zeta = e^{2\pi i/3}$. Pick respective representatives $[1:1:1]$, $[1:1:\zeta]$, $[1:\zeta:\zeta]$, and $[1:0:0]$ of Z_1, Z_2, Z_3, and Z_0.

Lemma III.19. *Every degenerate Sklyanin algebra is a Zhang twist of one the following algebras: $S(1,1,1)$, $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, and $S(1,0,0)$.*

Proof. Consider the following graded automorphisms of the degenerate Sklyanin algebras $S(a,b,c)$:

$$\sigma : \{x \mapsto \zeta x,\ y \mapsto \zeta^2 y,\ z \mapsto z\} \text{ and } \tau : \{x \mapsto y,\ y \mapsto z,\ z \mapsto x\}.$$

Now a routine computation shows that σ and τ yield the Zhang twists:

$$
\begin{array}{lll}
S(1,1,1)^\sigma = S(1,\zeta,\zeta^2), & S(1,1,1)^{\sigma^{-1}} = S(1,\zeta^2,\zeta) & \text{for } Z_1; \\
S(1,1,\zeta)^\sigma = S(1,\zeta,1), & S(1,1,\zeta)^{\sigma^{-1}} = S(1,\zeta^2,\zeta^2) & \text{for } Z_2; \\
S(1,\zeta,\zeta)^\sigma = S(1,\zeta^2,1), & S(1,\zeta,\zeta)^{\sigma^{-1}} = S(1,1,\zeta^2) & \text{for } Z_3; \\
S(1,0,0)^\tau = S(0,1,0), & S(1,0,0)^{\tau^{-1}} = S(0,0,1) & \text{for } Z_0.
\end{array}
$$

For instance, consider the Zhang twist $S(1,1,1)^\sigma$. Now it has three quadratic relations of the form:

$$\{yz + \alpha zy + \beta x^2,\ zx + \alpha xz + \beta y^2,\ xy + \alpha yx + \beta z^2\},$$

where α, $\beta \in k$. Note that $yz + \alpha zy + \beta x^2$ is a relation of $S(1,1,1)^\sigma$ if and only if $y \cdot z^\sigma + \alpha z \cdot y^\sigma + \beta x \cdot x^\sigma$ is a relation of $S(1,1,1)$. This is equivalent to $yz + \alpha\zeta^2 zy + \beta\zeta x^2$ equal to zero in $S(1,1,1)$, which implies $\alpha = \zeta$ and $\beta = \zeta^2$. When we consider the other two relations of $S(1,1,1)^\sigma$, we also get that $\alpha = \zeta$ and $\beta = \zeta^2$. Thus $S(1,1,1)^\sigma = S(1,\zeta,\zeta^2)$. $\square$

By Theorem III.18, it suffices to study a representative of each of the four classes of degenerate three-dimensional Sklyanin algebras.

Correction: The point scheme of S(1,1,1) is actually much larger than what is computed here. Please see the corrigendum to "Degenerate Sklyanin algebras and Generalized Twisted Homogeneous Coordinate rings" J. Alg (2009)

3.2.2 Computation of V_d for $S(1,1,1)$

We now compute the truncated point schemes of $S(1,1,1)$ in detail. Calculations for the other three representative degenerate Sklyanin algebras, $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, $S(1,0,0)$, will follow with similar reasoning. We first discuss how to build a truncated point module M' of length d, when provided with a truncated point module M of length $d-1$.

Let us explore the correspondence between truncated point modules and truncated point schemes. When given a truncated point module $M = \bigoplus_{i=0}^{d-1} M_i \in \Gamma_{d-1}$, multiplication from $S = S(a,b,c)$ is determined by a point $p = (p_0, \ldots, p_{d-2}) \in V_{d-1}$

(Definition III.14, Equation (3.4)) in the following manner. As M is cyclic, M_i has basis say $\{m_i\}$. Furthermore for $x, y, z \in S$ with
$p_i = [x_i : y_i : z_i] \in \mathbb{P}^2$, we get the left S-action on m_i determined by p_i:

$$\begin{aligned} x \cdot m_i &= x_i m_{i+1}, \quad x \cdot m_{d-1} = 0; \\ y \cdot m_i &= y_i m_{i+1}, \quad y \cdot m_{d-1} = 0; \\ z \cdot m_i &= z_i m_{i+1}, \quad z \cdot m_{d-1} = 0. \end{aligned} \tag{3.6}$$

Conversely given a point $p = (p_0, \ldots, p_{d-2}) \in V_{d-1}$, one can build a module $M \in \Gamma_{d-1}$ unique up to isomorphism by reversing the above process. We summarize this discussion in the following remark.

Remark III.20. Refer to notation from Lemma III.13. To construct $M' \in \Gamma_d$ from $M \in \Gamma_{d-1}$ associated to $p \in V_{d-1}$, we require $p_{d-1} \in \mathbb{P}^2$ such that $p' = (p, p_{d-1}) \in V_d$.

Now we begin to study the behavior of truncated point modules over S_{deg} through the examination of truncated point schemes in the next two lemmas.

Lemma III.21. *Let $p = (p_0, \ldots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \notin \mathfrak{D}$ (refer to (3.5)). Then for each i, there exists a unique $p_{d-1} \in Z_i$ so that $p' := (p, p_{d-1}) \in V_d$.*

Proof. For Z_1, we study the representative algebra $S(1,1,1)$. If such a p_{d-1} exists, then $f_{d-2} = g_{d-2} = h_{d-2} = 0$ so we would have

$$\mathbb{M}_{111,d-2} \cdot (x_{d-1}\ y_{d-1}\ z_{d-1})^T = 0$$

(Definition III.14, Equation (3.4)). Now we have that rank$(\mathbb{M}_{111,d-2}) \leq 2$ as

$$(x_{d-3}\ y_{d-3}\ z_{d-3}) \cdot \mathbb{M}_{111,d-2} = 0.$$

By an argument in the proof of Proposition II.15, we see that if rank$(\mathbb{M}_{111,d-2}) = 1$, then $p_{d-2} \in \mathfrak{D}$. Thus rank$(\mathbb{M}_{111,d-2}) = 2$ when $p_{d-2} \notin \mathfrak{D}$ and the tuple $(x_{d-1},\ y_{d-1},\ z_{d-1})$ is unique up to scalar multiple. Thus the point p_{d-1} is projectively unique.

To verify the existence of p_{d-1}, we solve the system of equations

$$
(3.7) \qquad \begin{aligned}
f_{d-2} : y_{d-1}z_{d-2} + z_{d-1}y_{d-2} + x_{d-1}x_{d-2} &= 0,\\
g_{d-2} : z_{d-1}x_{d-2} + x_{d-1}z_{d-2} + y_{d-1}y_{d-2} &= 0,\\
h_{d-2} : x_{d-1}y_{d-2} + y_{d-1}x_{d-2} + z_{d-1}z_{d-2} &= 0,\\
x_{d-2}^3 + y_{d-2}^3 + z_{d-2}^3 &= 3x_{d-2}y_{d-2}z_{d-2},\\
x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 &= 3x_{d-1}y_{d-1}z_{d-1}.
\end{aligned}
$$

Here, the last two equations come from the fact that p_{d-2}, $p_{d-1} \in E$ (Lemma III.16).

Furthermore with $\zeta = e^{2\pi i/3}$, the curve $E = E_{111}$ is the union of three projective lines:

$$
(3.8) \qquad \mathbb{P}^1_A : x = -(y+z),\ \mathbb{P}^1_B : x = -(\zeta^2 y + \zeta z),\ \mathbb{P}^1_C : x = -(\zeta y + \zeta^2 z)
$$

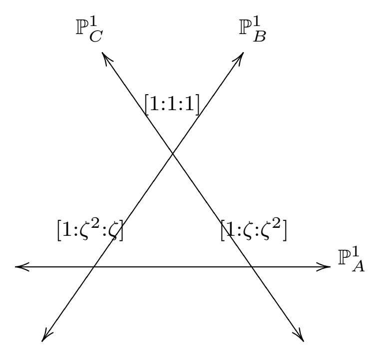

Figure 3.1: The curve $E = E_{111} \subseteq \mathbb{P}^2 :\ x^3 + y^3 + z^3 - 3xyz = 0$

Using the algebra software Maple, we have the following three solutions to (3.7) with $(p_{d-2}, p_{d-1}) \in (E \smallsetminus \mathfrak{D}) \times E$:

$$
\left\{ \begin{array}{l}
(p_{d-2} \in \mathbb{P}^1_A, \quad [1:1:1]),\\
(p_{d-2} \in \mathbb{P}^1_B, \quad [1:\zeta:\zeta^2]),\\
(p_{d-2} \in \mathbb{P}^1_C, \quad [1:\zeta^2:\zeta])
\end{array} \right\}.
$$

Thus when $p_{d-2} \notin \mathfrak{D}$, there exists an unique point $p_{d-1} \in Z_1$ so that $(p_0, \ldots, p_{d-2}, p_{d-1}) \in V_d$.

Now consider the following notation.

	$S(1,1,\zeta)$	$S(1,\zeta,\zeta)$
$\mathbb{P}^1_D$	$x=-(y+\zeta^2 z)$	$x=-(y+\zeta z)$
$\mathbb{P}^1_E$	$x=-(\zeta^2 y+z)$	$x=-(\zeta y+z)$
$\mathbb{P}^1_F$	$x=-\zeta(y+z)$	$x=-\zeta^2(y+z)$

For $S(1,1,\zeta)$, the uniqueness of p_{d-1} still holds when $p_{d-2}\notin\mathfrak{D}$ and the following are the solutions $(p_{d-2},p_{d-1})\in(E\smallsetminus\mathfrak{D})\times E$ satisfying $f_{d-2}=g_{d-2}=h_{d-2}=0$:

$$\left\{\begin{array}{ll}(p_{d-2}\in\mathbb{P}^1_D, & [1:1:\zeta]),\\ (p_{d-2}\in\mathbb{P}^1_E, & [1:\zeta:1]),\\ (p_{d-2}\in\mathbb{P}^1_F, & [1:\zeta^2:\zeta^2])\end{array}\right\}.$$

For $S(1,\zeta,\zeta)$, again the uniqueness of p_{d-1} holds when $p_{d-2}\notin\mathfrak{D}$ and we have similar set of solutions in $(E\smallsetminus\mathfrak{D})\times E$ for $f_{d-2}=g_{d-2}=h_{d-2}=0$:

$$\left\{\begin{array}{ll}(p_{d-2}\in\mathbb{P}^1_D, & [1:\zeta:\zeta]),\\ (p_{d-2}\in\mathbb{P}^1_E, & [1:1:\zeta^2]),\\ (p_{d-2}\in\mathbb{P}^1_F, & [1:\zeta^2:1])\end{array}\right\}.$$

For Z_0, we study the algebra $S(1,0,0)$. Now if $p_i\in E=E_{1,0,0}$ satisfies $f_{d-2}=g_{d-2}=h_{d-2}=0$, then we get the system of equations:

$$\begin{aligned} y_{d-1}z_{d-2} &= z_{d-1}x_{d-2} = x_{d-1}y_{d-2} = 0\\ x_{d-2}y_{d-2}z_{d-2} &= x_{d-1}y_{d-1}z_{d-1} = 0\end{aligned} \tag{3.9}$$

If $p_{d-2}=[0:y_{d-2}:z_{d-2}]\notin\mathfrak{D}$, then both y_{d-2} and $z_{d-2}\neq 0$. Hence $x_{d-1}=y_{d-1}=0$ and $p_{d-1}=[0:0:1]\in Z_0$. On the other hand if $p_{d-2}=[1:y_{d-2}:z_{d-2}]$, then $z_{d-1}=0$, i.e. $p_{d-1}=[x_{d-1}:y_{d-1}:0]$. Now since $p_{d-2}\notin\mathfrak{D}$, we have that either y_{d-2} or $z_{d-2}\neq 0$. In the first case, if $y_{d-2}\neq 0$, then $x_{d-1}=0$ and $p_{d-1}=[0:1:0]\in Z_0$. Otherwise $z_{d-2}\neq 0$ implies that $y_{d-1}=0$ so $p_{d-1}=[1:0:0]\in Z_0$.

Thus we have verified the lemma. $\square$

The next result explores the case when $p_{d-2} \in Z_i$ for $i = 1, 2, 3$.

Lemma III.22. *Let* $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ *with* $p_{d-2} \in Z_i$ *for* $i = 1, 2, 3$. *Then for any* $[y_{d-1} : z_{d-1}] \in \mathbb{P}^1$ *there exists a function* θ *of two variables so that*

$$p_{d-1} = [\theta(y_{d-1}, z_{d-1}) : y_{d-1} : z_{d-1}] \notin Z_i$$

which satisfies $(p_0, \dots, p_{d-2}, p_{d-1}) \in V_d$.

Proof. The point $p' = (p, p_{d-1}) \in V_d$ needs to satisfy $f_i = g_i = h_i = 0$ for $0 \le i \le d-2$ (Definition III.14). Since $p \in V_{d-1}$, we need only to consider the equations $f_{d-2} = g_{d-2} = h_{d-2} = 0$ with $p_{d-2} \in Z_i$.

We study $S(1,1,1)$ for Z_1 so the relevant system of equations is

$$\begin{aligned} f_{d-2} &: \ y_{d-1}z_{d-2} + z_{d-1}y_{d-2} + x_{d-1}x_{d-2} = 0 \\ g_{d-2} &: \ z_{d-1}x_{d-2} + x_{d-1}z_{d-2} + y_{d-1}y_{d-2} = 0 \\ h_{d-2} &: \ x_{d-1}y_{d-2} + y_{d-1}x_{d-2} + z_{d-1}z_{d-2} = 0. \end{aligned}$$

If $p_{d-2} = [1 : 1 : 1] \in Z_1$, then $x_{d-1} = -(y_{d-1} + z_{d-1})$ is required. On the other hand, if $p_{d-2} = [1 : \zeta : \zeta^2]$ or $[1 : \zeta^2 : \zeta]$, we require $x_{d-1} = -\zeta(y_{d-1} + \zeta z_{d-1})$ or $x_{d-1} = -\zeta(\zeta y_{d-1} + z_{d-1})$ respectively. Thus our function θ is defined as

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:1:1] \\ -(\zeta^2 y_{d-1} + \zeta z_{d-1}), & \text{if } p_{d-2} = [1:\zeta:\zeta^2] \\ -(\zeta y_{d-1} + \zeta^2 z_{d-1}), & \text{if } p_{d-2} = [1:\zeta^2:\zeta]. \end{cases}$$

The arguments for $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, and $S(1,0,0)$ yield similar results as we now describe.

For Z_2, consider the representative algebra $S(1,1,\zeta)$ with corresponding equations $f_{d-2} = g_{d-2} = h_{d-2} = 0$. In a similar fashion, we have that:

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + \zeta^2 z_{d-1}), & \text{if } p_{d-2} = [1:1:\zeta] \\ -(\zeta^2 y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:\zeta:1] \\ -\zeta(y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:\zeta^2:\zeta^2]. \end{cases}$$

For Z_3, provided $S(1,\zeta,\zeta)$ as our representative algebra, we have that:

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + \zeta z_{d-1}), & \text{if } p_{d-2} = [1:\zeta:\zeta] \\ -(\zeta y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:1:\zeta^2] \\ -\zeta^2(y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:\zeta^2:1]. \end{cases} \qquad \square$$

Lemma III.23. *Consider the algebra $S(1,0,0)$. We have that solutions (p_{d-2}, p_{d-1}) to (3.9), with $p_{d-2} \in Z_0$, take the following form:*

$$p_{d-1} = \begin{cases} [x_{d-1} : y_{d-1} : 0], & \textit{if } p_{d-2} = [1:0:0] \\ [0 : y_{d-1} : z_{d-1}], & \textit{if } p_{d-2} = [0:1:0] \\ [x_{d-1} : 0 : z_{d-1}], & \textit{if } p_{d-2} = [0:0:1] \end{cases} \qquad \square$$

Fix a pair $(S_{deg}, Z_i(S_{deg}))$. We now know if $p_{d-2} \notin \mathfrak{D}$, then from every truncated point module of length d over S_{deg} we can produce a unique truncated point module of length $d+1$. Otherwise if $p_{d-2} \in Z_i$, we get a $\mathbb{P}^1$ worth of length $d+1$ modules. We summarize this in the following statement.

Proposition III.24. *The parameter space of Γ_d over S_{deg} is isomorphic to the singular and nondisjoint union of*

$$\begin{cases} \textit{three copies of } (\mathbb{P}^1)^{\times \frac{d-1}{2}} \textit{ and three copies of } (\mathbb{P}^1)^{\times \frac{d+1}{2}}, & \textit{for } d \textit{ odd;} \\ \textit{six copies of } (\mathbb{P}^1)^{\times \frac{d}{2}}, & \textit{for } d \textit{ even.} \end{cases}$$

The proof of this result will be given in Proposition III.26 below, which will also yield a more detailed statement. We restrict our attention to $S(1,1,1)$ for reasoning mentioned in the proofs of Lemmas III.21 and III.22.

3.2.2.1. Parameterization of Γ_2

Recall that length 3 truncated point modules of Γ_2 are in bijective correspondence to points on $V_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ (Lemma III.15) and it is our goal to depict this truncated point scheme. By Lemma III.16, we know that $V_2 \subseteq E \times E$.

Now to calculate V_2, recall that Γ_2 consists of length 3 truncated point modules $M_{(3)} := M_0 \oplus M_1 \oplus M_2$ where M_i is a 1-dimensional k-vector space say with basis m_i. The module $M_{(3)}$ has action determined by $(p_0, p_1) \in V_2$ (Equation (3.6)). Moreover Lemmas III.21 and III.22 provide the precise conditions for (p_0, p_1) to lie in $E \times E$. Namely,

Lemma III.25. *Refer to (3.8) for notation. The set of length 3 truncated point modules Γ_2 is parametrized by the scheme $V_2 = \mathbb{V}(f_0, g_0, h_0)$ which is the union of the six subsets:*

$$\begin{array}{ll} \mathbb{P}^1_A \times [1:1:1]; & [1:1:1] \times \mathbb{P}^1_A; \\ \mathbb{P}^1_B \times [1:\zeta:\zeta^2]; & [1:\zeta:\zeta^2] \times \mathbb{P}^1_B; \\ \mathbb{P}^1_C \times [1:\zeta^2:\zeta]; & [1:\zeta^2:\zeta] \times \mathbb{P}^1_C. \end{array}$$

of $E \times E$. Thus Γ_2 is isomorphic to 6 copies of $\mathbb{P}^1$. □

3.2.2.2. Parameterization of Γ_d for general d

To illustrate the parametrization of Γ_d, we begin with a truncated point module $M_{(d+1)}$ of length $d+1$ corresponding to $(p_0, p_1, \dots, p_{d-1}) \in V_d \subseteq (\mathbb{P}^2)^{\times d}$. Due to Lemmas III.16, III.21, and III.22, we know that $(p_0, p_1, \dots, p_{d-1})$ belongs to either

$$\underbrace{(E \smallsetminus \mathfrak{D}) \times Z_1 \times (E \smallsetminus \mathfrak{D}) \times Z_1 \times \dots}_{d} \text{ or } \underbrace{Z_1 \times (E \smallsetminus \mathfrak{D}) \times Z_1 \times (E \smallsetminus \mathfrak{D}) \times \dots}_{d}$$

where Z_1 is defined in (3.5).

By adapting the notation of Lemma III.22, we get in the first case that the point

$(p_0, p_1, \ldots, p_{d-1})$ is of the form

$$([\theta(y_0, z_0) : y_0 : z_0],\ [1 : \omega : \omega^2],\ [\theta(y_2, z_2) : y_2 : z_2],\ [1 : \omega : \omega^2], \ldots) \in (\mathbb{P}^2)^{\times d}$$

where $\omega^3 = 1$ and $\theta(y, z) = -(\omega y + \omega^2 z)$. Thus in this case, the set of length d truncated point modules is parameterized by three copies of $(\mathbb{P}^1)^{\times \lceil d/2 \rceil}$ with coordinates $([y_0 : z_0], [y_2 : z_2], \ldots, [y_{2\lceil d/2 \rceil - 1} : z_{2\lceil d/2 \rceil - 1}])$.

In the second case $(p_0, p_1, \ldots, p_{d-1})$ takes the form

$$([1 : \omega : \omega^2],\ [\theta(y_1, z_1) : y_1 : z_1],\ [1 : \omega : \omega^2], [\theta(y_3, z_3) : y_3 : z_3], \ldots) \in (\mathbb{P}^2)^{\times d}$$

and the set of truncated point modules is parameterized with three copies of $(\mathbb{P}^1)^{\times \lfloor d/2 \rfloor}$ with coordinates $([y_1 : z_1], [y_3 : z_3], \ldots, [y_{2\lfloor d/2 \rfloor - 1} : z_{2\lfloor d/2 \rfloor - 1}])$.

In other words, we have now proved the next result.

Proposition III.26. *Refer to (3.8) for notation. For $d \geq 2$ the truncated point scheme V_d for $S(1,1,1)$ is equal to the union of the six subsets $\bigcup_{i=1}^{6} W_{d,i}$ of $(\mathbb{P}^2)^{\times d}$ where*

$$\begin{aligned}
W_{d,1} &= \mathbb{P}^1_A \times [1 : 1 : 1] \times \mathbb{P}^1_A \times [1 : 1 : 1] \times \ldots, \\
W_{d,2} &= [1 : 1 : 1] \times \mathbb{P}^1_A \times [1 : 1 : 1] \times \mathbb{P}^1_A \times \ldots, \\
W_{d,3} &= \mathbb{P}^1_B \times [1 : \zeta : \zeta^2] \times \mathbb{P}^1_B \times [1 : \zeta : \zeta^2] \times \ldots, \\
W_{d,4} &= [1 : \zeta : \zeta^2] \times \mathbb{P}^1_B \times [1 : \zeta : \zeta^2] \times \mathbb{P}^1_C \times \ldots, \\
W_{d,5} &= \mathbb{P}^1_C \times [1 : \zeta^2 : \zeta] \times \mathbb{P}^1_C \times [1 : \zeta^2 : \zeta] \times \ldots, \\
W_{d,6} &= [1 : \zeta^2 : \zeta] \times \mathbb{P}^1_C \times [1 : \zeta^2 : \zeta] \times \mathbb{P}^1_C \times \ldots.
\end{aligned}$$

□

As a consequence, we obtain the proof of Proposition III.24 for $S(1,1,1)$. By the next result, this holds for the remaining degenerate Sklyanin algebras.

Proposition III.27. *Consider the following notation.*

	$S(1,1,\zeta)$	$S(1,\zeta,\zeta)$	$S(1,0,0)$
$\mathbb{P}^1_D$	$x=-(y+\zeta^2 z)$	$x=-(y+\zeta z)$	$z=0$
$\mathbb{P}^1_E$	$x=-(\zeta^2 y+z)$	$x=-(\zeta y+z)$	$x=0$
$\mathbb{P}^1_F$	$x=-\zeta(y+z)$	$x=-\zeta^2(y+z)$	$y=0$
q_1	$[1:1:\zeta]$	$[1:\zeta:\zeta]$	$[1:0:0]$
q_2	$[1:\zeta:1]$	$[1:1:\zeta^2]$	$[0:1:0]$
q_3	$[1:\zeta^2:\zeta^2]$	$[1:\zeta^2:1]$	$[0:0:1]$

For $d\geq 2$, the d^{th} truncated point scheme V_d of the algebras $S(1,1,\zeta)$ and of $S(1,\zeta,\zeta)$ is the union of the six subsets $\bigcup_{i=1}^{6} W_{d,i}$ of $(\mathbb{P}^2)^{\times d}$ where

$$\begin{aligned}
W_{d,1} &= \mathbb{P}^1_D\times\{q_1\}\times\mathbb{P}^1_D\times\{q_1\}\times\dots,\\
W_{d,2} &= \{q_1\}\times\mathbb{P}^1_D\times\{q_1\}\times\mathbb{P}^1_D\times\dots,\\
W_{d,3} &= \mathbb{P}^1_E\times\{q_2\}\times\mathbb{P}^1_E\times\{q_2\}\times\dots,\\
W_{d,4} &= \{q_2\}\times\mathbb{P}^1_E\times\{q_2\}\times\mathbb{P}^1_E\times\dots,\\
W_{d,5} &= \mathbb{P}^1_F\times\{q_3\}\times\mathbb{P}^1_F\times\{q_3\}\times\dots,\\
W_{d,6} &= \{q_3\}\times\mathbb{P}^1_F\times\{q_3\}\times\mathbb{P}^1_F\times\dots.
\end{aligned}$$

Moreover the V_d for the algebra $S(1,0,0)$ are given as the union of the following subsets of $(\mathbb{P}^2)^{\times d}$:

$$\begin{aligned}
W_{d,1} &= \mathbb{P}^1_D\times\{q_2\}\times\mathbb{P}^1_E\times\{q_3\}\times\mathbb{P}^1_F\times\{q_1\}\times\mathbb{P}^1_D\times\dots,\\
W_{d,2} &= \{q_2\}\times\mathbb{P}^1_E\times\{q_3\}\times\mathbb{P}^1_F\times\{q_1\}\times\mathbb{P}^1_D\times\{q_2\}\times\dots,\\
W_{d,3} &= \mathbb{P}^1_E\times\{q_3\}\times\mathbb{P}^1_F\times\{q_1\}\times\mathbb{P}^1_D\times\{q_2\}\times\mathbb{P}^1_E\times\dots,\\
W_{d,4} &= \{q_3\}\times\mathbb{P}^1_F\times\{q_1\}\times\mathbb{P}^1_D\times\{q_2\}\times\mathbb{P}^1_E\times\{q_3\}\times\dots,\\
W_{d,5} &= \mathbb{P}^1_F\times\{q_1\}\times\mathbb{P}^1_D\times\{q_2\}\times\mathbb{P}^1_E\times\{q_3\}\times\mathbb{P}^1_F\times\dots,\\
W_{d,6} &= \{q_1\}\times\mathbb{P}^1_D\times\{q_2\}\times\mathbb{P}^1_E\times\{q_3\}\times\mathbb{P}^1_F\times\{q_1\}\times\dots.
\end{aligned}$$

Proof. This follows by a similar proof to that of Proposition III.26 and is left to the reader. □

We thank Karen Smith for suggesting the following elegant way of interpreting the point scheme of $S(1,1,1)$.

Remark III.28. We can provide an alternate geometric description of the point scheme of the Γ of $S(1,1,1)$. Let $G := \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = <\zeta, \sigma>$ where $\zeta = e^{2\pi i/3}$ and $\sigma^2 = 1$. We define a G-action on $\mathbb{P}^2 \times \mathbb{P}^2$ as follows:

$$\zeta([x:y:z],[u:v:w]) = ([x:\zeta^2 y:\zeta z],[u:\zeta v:\zeta^2 w])$$

$$\sigma([x:y:z],[u:v:w]) = ([u:v:w],[x:y:z])$$

Note that G stabilizes $E \times E$ and acts transitively on the $W_{2,i}$. We extend the action of G to $(\mathbb{P}^2 \times \mathbb{P}^2)^{\times\infty}$ diagonally. Now we interpret Γ as

$$\Gamma \;=\; \varprojlim V_d \;=\; \varprojlim V_{2d} \;=\; \varprojlim \bigcup_i W_{2d,i} \;=\; G \cdot (\mathbb{P}^1_A \times [1:1:1])^{\times\infty},$$

as sets.

3.3 Point parameter ring of $S(1,1,1)$

We now construct a graded associative algebra P from truncated point schemes of the degenerate Sklyanin algebra $S = S(1,1,1)$. The analogous result for the other degenerate Sklyanin algebras will follow in a similar fashion and we leave the details to the reader. As is true for the Sklyanin algebras themselves, it will be shown that this algebra P is a proper factor of $S(1,1,1)$ and its properties closely reflect those of $S(1,1,1)$. We will for example show that P is not right noetherian, nor a domain. In other words, we establish Theorem III.4.

The definition of the algebra P initially appears in [ATVdB90, §3]. In particular for $P = P(S(a,b,c))$, recall that we have projection maps $pr_{1,\dots,d-1}$ and $pr_{2,\dots,d}$ from

$(\mathbb{P}^2)^{\times d}$ to $(\mathbb{P}^2)^{\times d-1}$. Restrictions of these maps to the truncated point schemes $V_d \subseteq (\mathbb{P}^2)^{\times d}$ (Definition III.14) yield

$$pr_{1,\dots,d-1}(V_d) \subset V_{d-1} \text{ and } pr_{2,\dots,d}(V_d) \subset V_{d-1} \text{ for all } d.$$

Definition III.29. Given the above data, we expand on the definition from the introduction. The **point parameter ring** $P = P(S(a,b,c))$ is an associative $\mathbb{N}$-graded ring defined as follows. First $P_d = H^0(V_d, \mathcal{L}_d)$ where $\mathcal{L}_d$ is the restriction of invertible sheaf

$$pr_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \dots \otimes pr_d^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(1,\dots,1)$$

to V_d. The multiplication map $\mu_{i,j} : P_i \times P_j \to P_{i+j}$ is then defined by applying H^0 to the isomorphism

$$pr_{1,\dots,i}^*(\mathcal{L}_i) \otimes_{\mathcal{O}_{V_{i+j}}} pr_{i+1,\dots,i+j}^*(\mathcal{L}_j) \to \mathcal{L}_{i+j}.$$

We declare $P_0 = k$.

From now on, we restrict our attention to $P(S(a,b,c))$ with $[a:b:c] \in \mathfrak{D}$, or rather for S_{deg}. We will later see in Theorem III.36 that $P(S_{deg})$ is generated in degree one; thus S_{deg} surjects onto P as $(S_{deg})_1 \cong P_1$. To begin the analysis of P for $S(1,1,1)$, recall that $V_1 = \mathbb{P}^2$ so

$$P_1 = H^0(V_1, pr_1^*\mathcal{O}_{\mathbb{P}^2}(1)) = kx \oplus ky \oplus kz$$

where $[x:y:z]$ are the coordinates of $\mathbb{P}^2$. For $d \geq 2$ we will compute $\dim_k P_d$ and then proceed to the more difficult task of identifying the multiplication maps $\mu_{i,j} : P_i \times P_j \to P_{i+j}$. Before we get to specific calculations for $d \geq 2$, let us recall that the schemes V_d are realized as the union of six subsets $\{W_{d,i}\}_{i=1}^6$ of $(\mathbb{P}^2)^{\times d}$ described in Proposition III.26 and Equation (3.8). These subsets intersect nontrivially so that each V_d for $d \geq 2$ is singular. More precisely,

Remark III.30. A routine computation shows that the singular subset, $\mathrm{Sing}(V_d)$, consists of six points:

$$\begin{aligned}
v_{d,1} &:= ([1:1:1],\ [1:\zeta:\zeta^2],\ [1:1:1],\ [1:\zeta:\zeta^2],\dots) && \in W_{d,2}\cap W_{d,3},\\
v_{d,2} &:= ([1:1:1],\ [1:\zeta^2:\zeta],\ [1:1:1],\ [1:\zeta^2:\zeta],\dots) && \in W_{d,2}\cap W_{d,5},\\
v_{d,3} &:= ([1:\zeta:\zeta^2],\ [1:1:1],\ [1:\zeta:\zeta^2],\ [1:1:1],\dots) && \in W_{d,1}\cap W_{d,4},\\
v_{d,4} &:= ([1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2],\dots) && \in W_{d,3}\cap W_{d,4},\\
v_{d,5} &:= ([1:\zeta^2:\zeta],\ [1:1:1],\ [1:\zeta^2:\zeta],\ [1:1:1],\dots) && \in W_{d,1}\cap W_{d,6},\\
v_{d,6} &:= ([1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta],\dots) && \in W_{d,5}\cap W_{d,6}.
\end{aligned}$$

where $\zeta = e^{2\pi i/3}$.

3.3.1 Computing the dimension of P_d

Our objective in this section is to prove

Proposition III.31. *For $d \geq 1$, $\dim_k P_d = 3\left(2^{\lfloor\frac{d+1}{2}\rfloor} + 2^{\lceil\frac{d-1}{2}\rceil}\right) - 6$.*

We first point out a result of Künneth that will be used several times in the rest of the chapter.

Theorem III.32. *[BG06, A.10.37] (Künneth's Formula) Let X_1 and X_2 be varieties over k. Let $\mathcal{E}$ be a locally free sheaf on X_1 and let $\mathcal{F}$ be a coherent sheaf on X_2. Then we have that*

$$H^n(X_1 \times X_2, pr_1^*\mathcal{E} \otimes pr_2^*\mathcal{F}) \;\cong\; \bigoplus_{p+q=n} H^p(X_1,\mathcal{E}) \otimes H^q(X_2,\mathcal{F}).$$

For the rest of the section, let $\mathbf{1}$ denote a sequence of 1s of appropriate length. Now consider the normalization morphism $\pi : V_d' \to V_d$ where V_d' is the disjoint union of the six subsets $\{W_{d,i}\}_{i=1}^6$ mentioned in Proposition III.26. This map induces the following short exact sequence of sheaves on V_d:

$$0 \to \mathcal{O}_{V_d}(\mathbf{1}) \to (\pi_*\mathcal{O}_{V_d'})(\mathbf{1}) \to \mathcal{S}(\mathbf{1}) \to 0, \tag{3.10}$$

where $\mathcal{S}$ is the skyscraper sheaf whose support is $\mathrm{Sing}(V_d)$, that is $\mathcal{S} = \bigoplus_{k=1}^6 \mathcal{O}_{\{v_{d,k}\}}$.

Note that we have

$$H^0(V_d, (\pi_* \mathcal{O}_{V'_d})(\mathbf{1})) \underset{k-\text{v.s.}}{\cong} H^0(V'_d, \mathcal{O}_{V'_d}(\mathbf{1})) \tag{3.11}$$

since the normalization morphism is a finite map, which in turn is an affine map [Har77, Exercises II.5.17(b), III.4.1]. To complete the proof of the proposition, we make the following assertion:

Sublemma III.33. $H^1(V_d, \mathcal{O}_{V_d}(\mathbf{1})) = 0$.

Assuming that Sublemma III.33 holds, we get from (3.10) the following long exact sequence of cohomology:

$$\begin{aligned} 0 \to H^0(V_d, \mathcal{O}_{V_d}(\mathbf{1})) &\to H^0(V_d, (\pi_* \mathcal{O}_{V'_d})(\mathbf{1})) \\ &\to H^0(V_d, \mathcal{S}(\mathbf{1})) \;\to\; H^1(V_d, \mathcal{O}_{V_d}(\mathbf{1})) = 0. \end{aligned}$$

Thus, with writing $h^0(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L})$, (3.11) implies that

$$\begin{aligned} \dim_k P_d = h^0(\mathcal{O}_{V_d}(\mathbf{1})) &= h^0((\pi_* \mathcal{O}_{V'_d})(\mathbf{1})) - h^0(\mathcal{S}(\mathbf{1})) \\ &= h^0(\mathcal{O}_{V'_d}(\mathbf{1})) - h^0(\mathcal{S}(\mathbf{1})) \\ &= \textstyle\sum_{i=1}^6 h^0(\mathcal{O}_{W_{d,i}}(\mathbf{1})) - 6. \end{aligned}$$

Therefore applying Proposition III.24 and Theorem III.32 completes the proof of Proposition III.31. It now remains to verify Sublemma III.33.

Proof of Sublemma III.33: By the discussion above, it suffices to show that

$$\delta_d : H^0(V'_d, \mathcal{O}_{V'_d}(\mathbf{1})) \to H^0\left(\bigcup_{k=1}^{6} \{v_{d,k}\},\ \mathcal{S}(\mathbf{1})\right)$$

is surjective. Referring to the notation of Proposition III.26 and Remark III.30, we choose $v_{d,i} \in \mathrm{Supp}(\mathcal{S}(\mathbf{1}))$ and W_{d,k_i} containing $v_{d,i}$. This W_{d,k_i} contains precisely two points of $\mathrm{Supp}(\mathcal{S}(\mathbf{1}))$ and say the other is $v_{d,j}$ for $j \neq i$. After choosing a basis $\{t_i\}_{i=1}^6$

for the six-dimensional vector space $H^0(\mathcal{S}(\mathbf{1}))$ where $t_i(v_{d,j}) = \delta_{ij}$, we construct a preimage of each t_i. Since $\mathcal{O}_{W_{d,k_i}}(\mathbf{1})$ is a very ample sheaf, it separates points. In other words there exists $\tilde{s}_i \in H^0(\mathcal{O}_{W_{d,k_i}}(\mathbf{1}))$ such that $\tilde{s}_i(v_{d,j}) = \delta_{ij}$. Extend this section $\tilde{s}_i$ to $s_i \in H^0(\mathcal{O}_{V_d'}(\mathbf{1}))$ by declaring $s_i = \tilde{s}_i$ on W_{d,k_i} and $s_i = 0$ elsewhere. Thus $\delta_d(s_i) = t_i$ for all i and the map δ_d is surjective as desired. □

This concludes the proof of Proposition III.31.

Corollary III.34. *We have that P has exponential growth hence infinite GK dimension. We also get that P is not left or right noetherian.*

Proof. Recall from Proposition III.31 that $\dim_k P_d = 3\left(2^{\lceil \frac{d-1}{2} \rceil} + 2^{\lfloor \frac{d+1}{2} \rfloor} - 2\right)$, which is greater than $3 \cdot 2^{d/2}$ for $d >> 0$. Moreover

$$\sum_{i \le d} 3 \cdot 2^{i/2} \ge 3\left(2^{\lceil d/2 \rceil} - 1\right) =: s_d.$$

Since $\overline{\lim}_d\, s_d^{1/d} = \sqrt{2}$, we have that P has exponential growth. By Theorem II.30, the ring P is not left or right noetherian. □

We can also determine the Hilbert series of P.

Proposition III.35. $H_P(t) = \dfrac{(1+t^2)(1+2t)}{(1-2t^2)(1-t)}$.

Proof. Recall from Proposition III.31 that $\dim_k P_d = 3\left(2^{\lceil \frac{d-1}{2} \rceil} + 2^{\lfloor \frac{d+1}{2} \rfloor}\right) - 6$ for $d \ge 1$ and that $\dim_k P_0 = 1$. Thus

$$\begin{aligned} H_P(t) &= 1 + 3\left(\sum_{d \ge 1} 2^{\lceil \frac{d-1}{2} \rceil} t^d + \sum_{d \ge 1} 2^{\lfloor \frac{d+1}{2} \rfloor} t^d - 2\sum_{d \ge 1} t^d\right) \\ &= 1 + 3\left(t\sum_{d \ge 0} 2^{\lceil \frac{d}{2} \rceil} t^d + 2t\sum_{d \ge 0} 2^{\lfloor \frac{d}{2} \rfloor} t^d - 2t\sum_{d \ge 0} t^d\right). \end{aligned}$$

Consider generating functions $a(t) = \sum_{d \ge 0} a_d t^d$ and $b(t) = \sum_{d \ge 0} b_d t^d$ for the respective sequences $a_d = 2^{\lceil d/2 \rceil}$ and $b_d = 2^{\lfloor d/2 \rfloor}$. Elementary operations result in $a(t) = \frac{1+2t}{1-2t^2}$ and $b(t) = \frac{1+t}{1-2t^2}$. Hence

$$H_P(t) = 1 + 3\left[t\left(\frac{1+2t}{1-2t^2}\right) + 2t\left(\frac{1+t}{1-2t^2}\right) - 2t\left(\frac{1}{1-t}\right)\right] = \frac{(1+t^2)(1+2t)}{(1-2t^2)(1-t)}.$$

□

3.3.2 The multiplication maps $\mu_{ij} : P_i \times P_j \to P_{i+j}$

In this section we examine the multiplication of the point parameter ring P of $S(1,1,1)$. In particular, we show that the multiplication maps are surjective which results in the following theorem.

Theorem III.36. *The point parameter ring P of $S(1,1,1)$ is generated in degree one.*

With similar reasoning, the ring $P = P(S_{deg})$ is generated in degree one for all S_{deg}. We leave the details to the reader.

Proof. It suffices to prove that the multiplication maps $\mu_{d,1} : P_d \times P_1 \to P_{d+1}$ are surjective for $d \geq 1$. Recall from Definition III.29 that $\mu_{d,1} = H^0(m_d)$ where m_d is the isomorphism

$$m_d : \mathcal{O}_{V_d \times \mathbb{P}^2}(1, \dots, 1, 0) \otimes_{\mathcal{O}_{V_{d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(0, \dots, 0, 1) \to \mathcal{O}_{V_{d+1}}(1, \dots, 1).$$

To use the isomorphism m_d, we employ the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{V_d \times \mathbb{P}^2}(1, \dots, 1, 0) \otimes_{\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(0, \dots, 0, 1) \arrow[d] \arrow[dr, "t_d"] & \\
\mathcal{O}_{V_d \times \mathbb{P}^2}(1, \dots, 1, 0) \otimes_{\mathcal{O}_{V_{d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(0, \dots, 0, 1) \arrow[r, "m_d"] & \mathcal{O}_{V_{d+1}}(1, \dots, 1).
\end{tikzcd} \tag{3.12}$$

The source of t_d is isomorphic to $\mathcal{O}_{V_d \times \mathbb{P}^2}(1, \dots, 1)$ and the map t_d is given by restriction to V_{d+1}. Hence we have the short exact sequence

$$0 \longrightarrow \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1}) \longrightarrow \mathcal{O}_{V_d \times \mathbb{P}^2}(\mathbf{1}) \xrightarrow{t_d} \mathcal{O}_{V_{d+1}}(\mathbf{1}) \longrightarrow 0, \tag{3.13}$$

where $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}$ is the ideal sheaf of V_{d+1} defined in $V_d \times \mathbb{P}^2$. Since Theorem III.32 and Sublemma III.33 implies that $H^1(\mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1})) = 0$, the cokernel of $H^0(t_d)$ is $H^1\Big(\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})\Big)$. Now we assert:

Proposition III.37. $H^1\Big(\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ *for* $d \geq 1$.

By assuming that Proposition III.37 holds, we get the surjectivity of $H^0(t_d)$ for $d \geq 1$. Now by applying the global section functor to Diagram (3.12), we have that $H^0(m_d) = \mu_{d,1}$ is surjective for $d \geq 1$. This concludes the proof of Theorem III.36. $\square$

Proof of III.37. Consider the case $d = 1$. We study the ideal sheaf $\mathcal{I}_{\frac{V_2}{\mathbb{P}^2\times\mathbb{P}^2}} := \mathcal{I}_{V_2}$ by using the resolution of the ideal of defining relations (f_0, g_0, h_0) for V_2 (Equations (3.3)) in the $\mathbb{N}^2$-graded ring $R = k[x_0, y_0, z_0, x_1, y_1, z_1]$. Note that each of the defining equations have bidegree (1,1) in R and we get the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3,-3) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{I}_{V_2} \to 0.$$

Twisting the above sequence with $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{O}^{\oplus 3}_{\mathbb{P}^2\times\mathbb{P}^2} \xrightarrow{f} \mathcal{I}_{V_2}(1,1) \to 0.$$

Let $\mathcal{K} = \ker(f)$. Then $h^0(\mathcal{I}_{V_2}(1,1)) = 3 - h^0(\mathcal{K}) + h^1(\mathcal{K})$. On the other hand, $H^1(\mathcal{O}_{\mathbb{P}^2}(j)) = H^2(\mathcal{O}_{\mathbb{P}^2}(j)) = 0$ for $j = -1, -2$. Thus Theorem III.32 applied the cohomology of the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{K} \to 0$$

results in $h^0(\mathcal{K}) = h^1(\mathcal{K}) = 0$. Hence $h^0(\mathcal{I}_{V_2}(1,1)) = 3$.

Now using the long exact sequence of cohomology arising from the short exact sequence

$$0 \to \mathcal{I}_{V_2}(1,1) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1) \to \mathcal{O}_{V_2}(1,1) \to 0,$$

and the facts:

$$h^0(\mathcal{I}_{V_2}(1,1)) = 3, \qquad h^0(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 9$$
$$h^0(\mathcal{O}_{V_2}(1,1)) = \dim_k P_2 = 6, \qquad h^1(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 0$$

we conclude that $H^1(\mathcal{I}_{V_2}(1,1)) = 0$.

For $d \geq 2$ we will construct a commutative diagram to assist with the study of the cohomology of the ideal sheaf $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})$. Recall from (3.10) that we have the following normalization sequence for V_d:

$$0 \longrightarrow \mathcal{O}_{V_d} \longrightarrow \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}} \longrightarrow \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}} \longrightarrow 0. \qquad (\dagger_d)$$

Consider the sequence

$$pr^*_{1,\ldots,d}\big((\dagger_d)\otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(\mathbf{1})\big) \otimes_{\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}} pr^*_{d+1}\mathcal{O}_{\mathbb{P}^2}(1)$$

and its induced sequence of restrictions to V_{d+1}, namely

$$0 \to \mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to 0. \qquad (3.14)$$

Now $V_{d+1} \subseteq V_d \times \mathbb{P}^2$ and $(W_{d,i} \times \mathbb{P}^2) \cap V_{d+1} = W_{d+1,i}$ due to Proposition III.26 and Remark III.30. We also have that $(\{v_{d,k}\} \times \mathbb{P}^2) \cap V_{d+1} = \{v_{d+1,k}\}$ for all i,k. Therefore the sequence (3.14) is equal to $(\dagger_{d+1}) \otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(\mathbf{1})$. In other words, we are given the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{V_{d+1}}(\mathbf{1}) \arrow[r] & \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d+1,i}}(\mathbf{1}) \arrow[r] & \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d+1,k}\}}(\mathbf{1}) \arrow[r] & 0.
\end{tikzcd}$$

Diagram 1: Understanding $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(1,\ldots,1)$

where the vertical maps are given by restriction to V_{d+1}. Observe that the kernels of the vertical maps (from left to right) are respectively $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})$, $\bigoplus_i \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})$, and $\bigoplus_k \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})$, and the cokernels are all 0.

By the Sublemma III.33 and Theorem III.32, we have that

$$H^1(\mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1})) = H^1(\mathcal{O}_{V_{d+1}}(\mathbf{1})) = 0.$$

Hence the application of the global section functor to Diagram 1 yields Diagram 2 below.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^0\left(V_d \times \mathbb{P}^2, \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d,i} \times \mathbb{P}^2, \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i} \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[r, dashed, "\psi"] \arrow[d, "\alpha"'] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\} \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[d, "\lambda"] & \\
0 \arrow[r] & H^0\left(V_d \times \mathbb{P}^2, \mathcal{O}_{V_d \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d,i} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i} \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r, "\nu"] \arrow[d, "\beta"'] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{O}_{\{v_{d,k}\} \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0\left(V_{d+1}, \mathcal{O}_{V_{d+1}}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d+1,i}, \mathcal{O}_{W_{d+1,i}}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d+1,k}\}, \mathcal{O}_{\{v_{d+1,k}\}}(\mathbf{1})\right) \arrow[r] \arrow[d] & 0 \\
 & H^1\left(V_d \times \mathbb{P}^2, \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\right) & \bigoplus_{i=1}^{6} H^1\left(W_{d,i} \times \mathbb{P}^2, \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i} \times \mathbb{P}^2}}(\mathbf{1})\right) & \bigoplus_{k=1}^{6} H^1\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\} \times \mathbb{P}^2}}(\mathbf{1})\right) &
\end{tikzcd}
$$

Diagram 2: Induced Cohomology from Diagram 1

Now by the Snake Lemma, we get the following sequence:

$$\begin{aligned} \cdots \longrightarrow \bigoplus_{i=1}^{6} H^0\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\right) \xrightarrow{\psi} \bigoplus_{k=1}^{6} H^0\left(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\right) \\ \longrightarrow H^1\left(\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})\right) \longrightarrow \bigoplus_{i=1}^{6} H^1\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\right) \to \cdots. \end{aligned}$$

In Lemma III.38, we will show that $\bigoplus_i H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ for $d \geq 2$. Furthermore the surjectivity of the map ψ will follow from Lemma III.39. This will complete the proof of Proposition III.37.

Lemma III.38. $\bigoplus_{i=1}^{6} H^1\Big(W_{d,i} \times \mathbb{P}^2,\ \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ *for* $d \geq 2$.

Proof. We consider the different parities of d and i separately. For d even and i odd,

$$\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}} \;\cong\; \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(0,\ldots,0,-1)$$

because $W_{d+1,i}$ is defined in $W_{d,i} \times \mathbb{P}^2$ by one equation of degree $(0,\ldots,0,1)$ (Proposition III.26). Twisting by $\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(1,\ldots,1)$ results in

$$H^1\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(1,\ldots,1)\right) \cong H^1\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\ldots,1,0)\right). \tag{3.15}$$

Since $W_{d,i}$ is the product of $\mathbb{P}^1$ and points lying in $\mathbb{P}^2$ and $H^1(\mathcal{O}_{\mathbb{P}^1}(1)) = H^1(\mathcal{O}_{\{pt\}}(1)) = H^1(\mathcal{O}_{\mathbb{P}^2}) = 0$, Theorem III.32 implies that the right hand side of (3.15) is equal to zero.

Consider the case of d and i even. As $pr_{1,\ldots,d}(W_{d+1,i}) = W_{d,i}$ and $pr_{d+1}(W_{d+1,i}) = [1 : \omega : \omega^2]$ for $\omega = \omega_{d,i}$ a third of unity, we have that $W_{d+1,i}$ is defined in $W_{d,i} \times \mathbb{P}^2$ by two equations of degree $(0,\ldots,0,1)$. The defining equations (in variables x, y, z) of $[1 : \omega : \omega^2]$ form a $k[x,y,z]$-regular sequence and so we have the Koszul resolution of $\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}} \otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(1,\ldots,1)$:

$$0 \to \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\ldots,1,-1) \to \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\ldots,1,0)^{\oplus 2} \tag{3.16}$$
$$\to \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(1,\ldots,1) \to 0.$$

Now apply the global section functor to sequence (3.16) and note that

$$H^j(\mathcal{O}_{W_{d,i}}(1,\ldots,1)) = H^j(\mathcal{O}_{\mathbb{P}^2}) = H^j(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0 \text{ for } j = 1,2.$$

Hence Theorem III.32 yields

$$H^1\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\ldots,1,0)\right)^{\oplus 2} = H^2\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\ldots,1,-1)\right) = 0.$$

Therefore $H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ for d and i even.

We conclude that for d even, we know $\bigoplus_{i=1}^{6} H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$. For d odd, the same conclusion is drawn by swapping the arguments for the i even and i odd subcases. □

Lemma III.39. *The map ψ is surjective for $d \geq 2$.*

Proof. Refer to the notation from Diagram 2. To show ψ is onto, here is our plan of attack.

1. Choose a basis of $\bigoplus_k H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ so that each basis element t lies in $H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k_0}\}}{\{v_{d,k_0}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ for some $k = k_0$. For such a basis element t, identify its image under λ in $\bigoplus_k H^0\left(\mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1})\right)$.

2. Construct for $\lambda(t)$ a suitable preimage $s \in \nu^{-1}(\lambda(t))$.

3. Prove $s \in \ker(\beta)$.

As a consequence, s lies in $\bigoplus_i H^0\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ and serves as a preimage to t under ψ. In other words, ψ is surjective. To begin, fix such a basis element t and integer k_0.

Step 1: Observe that $pr_{1,\ldots,d}(\{v_{d+1,k_0}\}) = \{v_{d,k_0}\}$ and $pr_{d+1}(\{v_{d+1,k_0}\}) = [1:\omega:\omega^2]$ for some ω, a third root of unity (Remark III.30). Thus our basis element $t \in \bigoplus_k H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ is of the form

$$t = a(\omega x_d - y_d) + b(\omega^2 x_d - z_d) \tag{3.17}$$

for some $a, b \in k$, with $\{\omega x_d - y_d,\ \omega^2 x_d - z_d\}$ defining $[1:\omega:\omega^2]$ in the $(d+1)^{st}$ copy of $\mathbb{P}^2$. Note that λ is the inclusion map so we may refer to $\lambda(t)$ as t. This concludes Step 1. □

Step 2: Next we construct a suitable preimage $s \in \nu^{-1}(\lambda(t))$. Referring to Remark III.30, let us observe that for all k, there is an unique even integer $:= i''_k$ and unique odd integer $:= i'_k$ so that $v_{d,k} \in W_{d,i''_k} \cap W_{d,i'_k}$ for all $k = 1, \ldots, 6$. For instance with $k_0 = 1$, we consider the membership $v_{d,1} \in W_{d,2} \cap W_{d,3}$; hence $i''_1 = 2$ and $i'_1 = 3$.

As a consequence, $\lambda(t)$ has preimages under ν in

$$H^0\left(W_{d,i''_{k_0}} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i''_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\right) \oplus\ H^0\left(W_{d,i'_{k_0}} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i'_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\right).$$

For d even (respectively odd) we write $i_{k_0} := i''_{k_0}$ (respectively $i_{k_0} := i'_{k_0}$). Therefore we intend to construct $s \in \nu^{-1}(t)$ belonging to $H^0\Big(\mathcal{O}_{W_{d,i_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\Big)$. However this $W_{d,i_{k_0}}$ will also contain another point $v_{d,j}$ for some $j \neq k_0$. Let us define the global section $\tilde{s} \in H^0\Big(\mathcal{O}_{W_{d,i_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\Big)$ as follows. Since $\mathcal{O}_{W_{d,i_{k_0}}}(\mathbf{1})$ is a very ample sheaf, we have a global section $\tilde{s}_{k_0}$ separating the points v_{d,k_0} and $v_{d,j}$; say $\tilde{s}_{k_0}(v_{d,k}) = \delta_{k_0,k}$. We then use (3.17) to define $\tilde{s}$ by

$$\tilde{s} = \tilde{s}_{k_0} \cdot [a(\omega x_d - y_d) + b(\omega^2 x_d - z_d)].$$

where $[1:\omega:\omega^2] = pr_{d+1}(\{v_{d+1,k_0}\})$. We now extend this section $\tilde{s}$ to

$$s \in \bigoplus_{i=1}^{6} H^0\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\right) \cong \left(\bigoplus_{i=1}^{6} H^0\left(\mathcal{O}_{W_{d,i}}(\mathbf{1})\right)\right) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)).$$

This is achieved by setting $s = \tilde{s}$ on $W_{d,i_{k_0}} \times \mathbb{P}^2$ and 0 elsewhere. To check that $\nu(s) = t$, note

$$s = \bigoplus_{i=1}^{6} s_i \text{ where } s_i \in H^0\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\right), \; s_i = \begin{cases} \tilde{s}, & i = i_{k_0}, \\ \\ 0, & i \neq i_{k_0}; \end{cases} \tag{3.18}$$

Therefore by the construction of $\tilde{s}$, we have $\nu(\tilde{s}) = t\big|_{\{v_{d,k_0}\}\times\mathbb{P}^2}$. Hence we have built our desired preimage $s \in \nu^{-1}(t)$ and this concludes Step 2. □

Step 3: Recall the structure of s from (3.18). By definition of β, we have that $\beta(s) = \beta\left(\bigoplus_{i=1}^6 s_i\right)$ is equal to $\bigoplus_{i=1}^6 \left(s_i\big|_{W_{d+1,i}}\right)$.

For $i \neq i_{k_0}$, we clearly get that $s_i\big|_{W_{d+1,i}} = 0$. On the other hand, the key point of our construction is that $W_{d+1,i_{k_0}} = W_{d,i_{k_0}} \times [1 : \epsilon : \epsilon^2]$ for some $\epsilon^3 = 1$ as i_{k_0} is chosen to be even (respectively odd) when d is even (respectively odd) (Proposition III.26). Moreover $v_{d+1,k_0} \in W_{d+1,i_{k_0}}$ and

$$pr_{d+1}(W_{d+1,i_{k_0}}) = pr_{d+1}(\{v_{d+1,k_0}\}) = [1 : \omega : \omega^2]$$

where ω is defined by Step 1 and Remark III.30. Thus $\epsilon = \omega$. Now we have

$$s_{i_{k_0}}\big|_{W_{d+1,i_{k_0}}} = \tilde{s}_{k_0} \cdot [a(\omega x_d - y_d) + b(\omega^2 x_d - z_d)]\big|_{[1:\omega:\omega^2]} = 0.$$

Therefore $s_i\big|_{W_{d+1,i}} = 0$ for all $i = 1, \ldots, 6$. Hence $\beta(s) = 0$. □

Hence Steps 1-3 are complete which concludes the proof of Lemma III.39. □

Consequently, we have verified Proposition III.37. □

One of the main results why twisted homogeneous coordinate rings are so useful for studying Sklyanin algebras is that the former are factors of their corresponding Sklyanin algebra (by some homogeneous element; refer to Theorem II.16). The following corollaries to Theorem III.36 illustrate an analogous result for S_{deg}.

Corollary III.40. *Let P be the point parameter ring of a degenerate Sklyanin algebra S_{deg}. Then $P \cong S_{deg}/K$ for some ideal K of S_{deg} that has six generators of degree 4 and possibly higher degree generators.*

Proof. Since $(S_{deg})_1 \cong P_1$, we have by Theorem III.36 that S_{deg} surjects onto P say with kernel K. By Lemma III.8 we have that $\dim_k S_4 = 57$, yet we know $\dim_k P_4 = 63$ by Proposition III.31. Hence $\dim_k K_4 = 6$. The same results also imply that $\dim_k S_d = \dim_k P_d$ for $d \leq 3$. □

Corollary III.41. *The ring $P = P(S_{deg})$ is neither a domain or Koszul.*

Proof. By Corollary III.9, there exist linear nonzero elements $u, v \in S$ with $uv = 0$. The image of u and v are nonzero, hence P is not a domain due to Corollary III.40. Since P has degree 4 relations, it does not possess the Koszul property. □

3.4 Open problems pertaining to point parameter rings

This section is dedicated to furthering the study of noncommutative coordinate rings in noncommutative projective algebraic geometry. In particular, we outline open problems pertaining to the point parameter rings introduced in Definition I.9 (or in Definition III.29 for three-dimensional Sklyanin algebras). This is with a view towards a theory that encompasses all coordinate rings arising from the geometric data of noncommutative connected graded algebras.

The problems discussed in the section are prompted by the observation that one can produce point scheme data (Definition-Lemma II.6) and subsequently a point parameter ring for *any* connected graded ring. Recall that point parameter rings are thought of as generalizations of twisted homogeneous coordinate rings. Furthermore the catalyst for the first task is the relationship between three-dimensional Sklyanin algebras, $Skly_3$, and twisted homogeneous coordinate rings. Namely Theorem II.16

states that the point scheme data of a generic $Skly_3$ stabilizes to a smooth cubic curve E, and there is a ring surjection from this $Skly_3$ to a twisted homogeneous coordinate ring $B = B(E)$ with kernel generated by the degree 3 element g. See Figure 2.8 for a depiction.

Question III.42. *Let A be a connected graded ring for which there does not exist a corresponding twisted homogeneous coordinate ring. In other words, the point scheme data of A does not stabilize to a projective scheme. Determine conditions on A so that there is a ring surjection from A onto its point parameter ring $P(A)$.*

Now the techniques of [ATVdB90], specifically the construction of point scheme data and noncommutative coordinate rings, is typically applied to strongly noetherian rings.

Definition III.43. A k-algebra A is said to be **strongly (left) noetherian** if $A \otimes_k C$ is a left noetherian ring for all commutative noetherian k-algebras C.

In fact, we have a result reminiscent of Theorem II.16 for all strongly noetherian connected graded k-algebras.

Theorem III.44. *[AZ01, Corollaries E4.11,E4.12] [RZ08, Theorem 1.1] Let A be a strongly noetherian connected graded k-algebra, which is generated in degree one. Then, we have that the following statements hold.*

1. *The point scheme X of A exists.*

2. *There is a ring homomorphism from A to a canonically constructed twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$, which is surjective in high degree. Moreover, the sheaf $\mathcal{L}$ is σ-ample and the kernel of this map is known.*

Only recently have these methods been extended to a class of non-strongly noetherian rings, so-called *naïve blowups* [KRS05]. Such rings are in fact noetherian and serve as examples in the ongoing program of classifying noncommutative surfaces. Meanwhile, more peculiar algebras have recently been discovered in [RS] and essentially nothing is known about their geometry. This prompts the following project.

Question III.45. *(Toby Stafford) Formulate a theory of noncommutative coordinate rings that encapsulates results on twisted homogeneous coordinate rings, point parameter rings, and naïve blowup rings constructed from the geometric data of a given connected graded noetherian algebra. Secondly, apply this theory to establish geometric structure for the rings of [RS].*

This question is partially answered in recent work of Nevins and Sierra.

Theorem III.46. *[NSa] Under several technical conditions we have that if A is a noetherian connected graded k-algebra, generated in degree one, then there exists a naïve blowup algebra S for which A surjects onto S in high degree.*

It is also conjectured that Question III.42 should speak to a class of noncommutative graded rings properly containing the noetherian structures. Namely we consider the class of *coherent* rings, which is motivated by the recent work on *coherent noncommutative algebraic geometry* [Pio08, Pol05]. Their results thus far include the classification of noncommutative projective lines and the definition of a coordinate ring of a noncommutative coherent projective scheme (Definition III.51, Theorem III.54). Details are found in the next section.

Question III.47. *(Lance Small, Paul Smith) Determine if the rings S_{deg} and $P(S_{deg})$ are coherent. If so, then determine whether $P(S_{deg})$ arises as a coordinate ring of a noncommutative coherent projective scheme in the sense of [Pio08].*

We show in the next section that the degenerate Sklyanin algebras are indeed coherent, yet it is not known if this is true for $P(S_{deg})$.

3.5 On coherent noncommutative algebraic geometry

This section further addresses Question III.47 in the previous section. First we introduce Piontkovskii-Polishchuk's theory of *coherent noncommutative algebraic geometry*, beginning with the definition, some properties, and examples of coherent (graded) algebras. We then discuss the notion of projective schemes in terms of coherent noncommutative algebraic geometry. Lastly we prove that degenerate Sklyanin algebras (Definition III.2) are coherent (Proposition III.55), and provide some consequences afterward.

Definition III.48. [Lam99, §4G]

1. A finitely generated left module M is **coherent** if every finitely generated submodule is finitely presented. Analogously a finitely generated graded module is **graded coherent** if every graded finitely generated submodule is finitely presented.

2. A (graded) algebra A is **left (graded) coherent** if every (homogeneous) finitely generated left-sided ideal in A is finitely presented, i.e. A is a (graded) coherent as a left module over itself.

3. We have analogous definitions for right coherence. Moreover an algebra is **coherent** if it is both left and right coherent.

Examples of (left) coherent rings include (left) noetherian rings and (left) semihereditary rings [Lam99, Example 4.46]. Moreover we have that the free algebras (on finitely many generators) are coherent [Pol05, Corollary 3.2].

Here are a couple of results that will help determine the coherence of a ring.

Lemma III.49. *[Lam99, Example 4.61(c)] Let I is a homogeneous ideal of a ring A,* *which is finitely generated as a left (right) ideal. If A is left (right) graded coherent, then A/I is left (right) coherent.*

Lemma III.50. *[Pio08, Proposition 3.2] Let $B = A/I$ where the algebra B is left (right) Noetherian and the ideal I is free as a right (left) A-module. Then the algebra A is left (right) graded coherent.*

Motivated by Artin and Zhang's approach to (noetherian) noncommutative algebraic geometry [AZ94], namely their construction of noncommutative schemes (see §2.1.5), Piontkovskii developed a version of a noncommutative projective scheme for the coherent setting [Pio08].

Definition III.51. [Pio08, §1] Provided a coherent algebra A, a **coherent projective scheme** is given by the abstract triple

$$(A - \text{c.qgr}, \mathcal{A}, s),$$

where A-c.qgr is the quotient category of finitely presented (=graded coherent) left graded A-modules by finite-dimensional modules, $\mathcal{A}$ is the image of A in A-c.qgr, and s is the autoequivalence of A-c.qgr induced by shift of grading.

The first non-trivial examples of this theory, namely coherent noncommutative projective lines, are given below.

Definition III.52. A graded algebra A is **regular** if it has finite global dimension d, and $\text{Ext}^q(k, A) = \delta_{d,q} \cdot k$.

Theorem III.53. *[Zha98, Theorem 0.1] A connected graded algebra is regular of global dimension 2 if and only if $A \cong k\{x_1, \dots, x_n\}/(b)$ with:*

1. $n \geq 2$;

2. *If the x_i's are labeled so that $1 \leq deg\ x_1 \leq \cdots \leq deg\ x_n$, then $deg\ x_i + deg\ x_{n-i}$ is a constant for all i,*

3. *There is a graded algebra automorphism α of $k\{x_1, \ldots, x_n\}$ so that*

$$b = \sum_{i=1}^{n} x_i \alpha(x_{n-i}).$$

In fact, such algebras are noetherian if and only if $n = 2$ [Zha98, Corollary 1.2]. Piontkovskii proves that all of these algebras are graded coherent [Pio08, Theorem 4.3]. Now we consider the algebras of Theorem III.53 that are generated in degree one.

Theorem III.54. *[Pio08, Theorem 1.4, Proposition 1.5] The coherent noncommutative $\mathbb{P}^1$ are classified. These are precisely the regular algebras, finitely generated in degree one, of global dimension 2.*

We aim to include degenerate Sklyanin algebras and eventually point parameter rings $P(A)$ (for arbitrary connected graded algebras A) into the theory of coherent noncommutative algebraic geometry. The following result is the first step towards this goal.

Proposition III.55. *The degenerate Sklyanin algebras are graded coherent.*

Proof. Recall that $S(a, b, c)$ from Definition III.1 is a degenerate Sklyanin algebra when

$$[a : b : c] \in \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\} \ \cup \ \{[a : b : c] \mid a^3 = b^3 = c^3 = 1\}.$$

In the first three cases, S is a monomial algebra defined by finitely many homogeneous relations. Thus by [Pio96], we have the result for these cases.

Now assume that $a^3 = b^3 = c^3 = 1$. Since $S(a,b,c) \cong S\left(1, \frac{b}{a}, \frac{c}{a}\right)$, we take $a = 1$ without loss of generality. Furthermore recall from Proposition III.7 that

$$S(1,b,c) \cong \frac{k\{x,y\}[z;\alpha,\delta]}{(\Omega)}$$

where Ω is a normal element of $k\{x,y\}[z;\alpha,\delta]$. Now by Lemma III.49, it suffices to show that $k\{x,y\}[z;\alpha,\delta]$ is (left) graded coherent.

Let $A := k\{x,y\}[z;\alpha,\delta]$, let $I = AxA + AyA$, and $B := k[z]$. Note that B is left noetherian. Moreover as

$$zx = -bxz - cy^2 \quad \text{and} \quad zy = -b^2yz - b^2cx^2,$$

one sees $I = xA + yA$ which is free as a right A-module. Therefore A is left graded coherent due to Lemma III.50. A similar argument also yields right coherence. □

To illustrate further consequences, we refer to [Lam99, §4F-4H] for results about the flatness of modules over S_{deg}. In particular, we have the following result of Chase.

Theorem III.56. *[Cha60, Theorem 2.1] A ring A is left coherent if and only if the direct product of any family of flat right A-modules is flat.*

One may compare this to a result of Papp and Bass: A ring A is left noetherian if and only if the direct sum of any family of injective left A-modules is injective [GW04, Theorem 5.23].

We also point out that there is work in progress with S. Paul Smith in realizing the category S_{deg}-qgr as equivalent to a module category over a direct limit of finite-dimensional algebras [SW].

CHAPTER IV

Representation theory of deformed Sklyanin algebras I: On the three-dimensional Sklyanin algebras $Skly_3$

This chapter and Chapter 5 present results towards the classification of irreducible finite-dimensional representations of so-called deformed Sklyanin algebras (Definition I.12=IV.2). Such a task is motivated by current research in string theory, namely the study of deformations of the $N = 4$ *super Yang-Mills theory in four dimensions*. We begin by introducing the physics background relevant to this thesis. We will see in particular that deformed Sklyanin algebras arise as *superpotential algebras*. Now let us remind the reader of the definition of three-dimensional Sklyanin algebras.

Definition IV.1. The **three-dimensional Sklyanin algebras**, denoted by $S(a,b,c)$ or $Skly_3$, are generated by three noncommuting variables x, y, z, subject to three relations:

$$ayz + bzy + cx^2 = 0$$
$$azx + bxz + cy^2 = 0$$
$$axy + byx + cz^2 = 0$$

for $[a:b:c] \in \mathbb{P}^2_k \setminus \mathfrak{D}$ where

$$\mathfrak{D} = \{[0:0:1],[0:1:0],[1:0:0]\} \ \cup \ \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

We then define the deformations of $Skly_3$ that are the focus of this chapter.

Definition IV.2. For $i = 1, 2, 3$, let a, b, c, d_i, e_i be scalars in k with $[a : b : c] \notin \mathfrak{D}$ (Definition IV.1). The **deformed Sklyanin algebras**, S_{def}, are generated by three noncommuting variables x, y, z, subject to three relations:

$$ayz + bzy + cx^2 + d_1x + e_1 = 0$$

$$azx + bxz + cy^2 + d_2y + e_2 = 0$$

$$axy + byx + cz^2 + d_3z + e_3 = 0.$$

In fact the name is motivated by the observation that, for (almost all) parameters (a, b, c), the ring S_{def} is a PBW-deformation is $Skly_3$ [BT07][EG, §3].

As we will explain in §4.2, the study of of the representation theory of S_{def} falls into two tasks, one of which is such a study for $Skly_3$. Hence the last section of this chapter is dedicated to the classification of irreducible finite-dimensional representations of three-dimensional Sklyanin algebras.

4.1 Physical Motivation

Here we discuss the physical motivation behind the task of classifying simple finite-dimensional S_{def}-modules. In particular we point out connections to work of Berenstein, Jejjala, and Leigh in the field of supersymmetric string theory. To introduce string theory in general, we use Zwiebach's description of the investigation of open strings in a system of M *D-branes* [Zwi09, §§1.3, 15.5].

> "Just as the strings of a violin are held stretched by pegs, the D-branes hold fixed the endpoints of the open strings whose lowest vibrational modes could represent the particles of the *Standard Model*."

Furthermore, we see that the interaction of open strings in the 'world' of M D-branes is in accordance with a certain *Yang-Mills theory*:

"On the world volume of M coincident D-branes, there are $U(M)$ *gauge fields* [...] whose low energy dynamics is governed by a Yang-Mills theory with *gauge group* $U(M)$."

Now the work in this chapter is prompted by the article [BJL00], in which the authors study various deformations of a *super* Yang-Mills theory with gauge group $U(M)$. More precisely the vacua, or solutions of *F-term constraints*, of these deformed theories are desired. Algebraically, this boils down to finding $M \times M$ matrix solutions to cyclic derivatives of a superpotential.

Definition IV.3. [EG, §3.1] Let V be a k-vector space with basis $x_1, \ldots, x_n$ and let $F = T(V) = k\{x_1, \ldots, x_n\}$ be the corresponding free algebra. The commutator quotient space $F_{cyc} = F/[F, F]$ is a k-vector space with the natural basis formed by cyclic words in the alphabet $x_1, \ldots, x_n$. Elements of F_{cyc} are referred to as **superpotentials** (or as **potentials** in some articles).

Let $\Phi = \sum_{\{i_1, i_2, \ldots, i_r\} \subset I} x_{i_1} x_{i_2} \cdots x_{i_r} \in F_{cyc}$ for some indexing set I. For each $j = 1, \ldots, n$, one defines $\partial_j \Phi \in F$, the corresponding partial derivative of Φ given by:

$$\partial_j \Phi = \sum_{\{s | i_s = j\}} x_{i_s+1} x_{i_s+2} \cdots x_{i_r} x_{i_1} x_{i_2} \cdots x_{i_s-1} \ \in F.$$

The elements $\partial_j \Phi$ are called **cyclic derivatives** of Φ.

The algebra $F/\left(\partial_j \Phi\right)_{j=1,\ldots,n}$ is called a **superpotential algebra**.

Example IV.4. Consider $F = k\{x, y, z\}$ with the superpotential $\Phi = xyz - xzy$. We get that

$$\partial_x \Phi = yz - zy, \ \partial_y \Phi = zx - xz, \ \partial_z \Phi = xy - yx.$$

Hence the polynomial algebra $k[x, y, z]$ arises as the superpotential algebra $F/(\partial \Phi)$.

We return to string theory with the following remark.

Remark IV.5. The superpotential relevant to the work of Berenstein et al. (after rescaling parameters) is $\Phi = \Phi_{marg} + \Phi_{rel}$ where:

$$\begin{aligned} \Phi_{marg} &= axyz + byxz + \tfrac{c}{3}(x^3 + y^3 + z^3), \\ \Phi_{rel} &= \tfrac{d_1}{2}x^2 + \tfrac{d_2}{2}(y^2 + z^2) + (e_1x + e_2y + e_3z). \end{aligned}$$

Here the two superpotentials respectively correspond to the *moduli spaces of marginal and relevant deformations* of a N=4 super Yang-Mills theory.

Observe that if $[a : b : c] \notin \mathfrak{D}$, then the corresponding superpotential algebra is precisely a deformed Sklyanin algebra with parameters $d_2 = d_3$. Hence the classification of M-dimensional representations of S_{def}, or more specifically the study of simple finite-dimensional S_{def}-modules, has physical implications.

4.2 Strategy to classify simple finite-dimensional modules over S_{def}

Let us consider the following notation.

Notation. Given an algebra A, let $\mathrm{Repr}_{<\infty}(A)$ be the set of isomorphism classes of finite-dimensional left A-modules and $\mathrm{Repr}_m(A)$ be the objects of dimension m. We denote by $\mathrm{Simp}_{<\infty}(A)$ and $\mathrm{Simp}_m(A)$, the respective subsets of simple modules.

We then formulate our aim as follows.

Classify the simple finite-dimensional modules over S_{def}.

The strategy to achieving the goal is as follows. First, homogenize the relations of S_{def} with a central element w. Then, study the representation theory of the resulting *central extension* D of $Skly_3$. Reasons for this construction are made clear in Proposition IV.7 below.

Definition IV.6. For $i = 1, 2, 3$, let a, b, c, d_i, e_i be scalars in k with $[a : b : c] \notin \mathfrak{D}$ (Definition IV.1). The **central extension** D **of** $Skly_3$ is generated by three variables

x, y, z and a central element w, subject to the relations:

$$ayz + bzy + cx^2 + d_1xw + e_1w^2 = 0$$

$$azx + bxz + cy^2 + d_2yw + e_2w^2 = 0$$

$$axy + byx + cz^2 + d_3zw + e_3w^2 = 0$$

$$xw - wx \;=\; yw - wy \;=\; zw - wz \;=\; 0.$$

Proposition IV.7. *Simple finite-dimensional modules over $D(a,b,c,d_i,e_i)$ are precisely those over $Skly_3(a,b,c)$ or over $S_{def}(a,b,c,d_i,e_i)$.*

In other words, our objective can be reformulated in terms of studying the simple finite-dimensional modules over the graded algebras $Skly_3$ and D.

Proof. For one direction, consider the surjection $\pi : D \twoheadrightarrow D/D(w-\lambda)$ for $\lambda \in k$. If $\lambda = 0$, then $D/D(w-\lambda) \cong Skly_3$. Otherwise we can rescale to assume that $\lambda = 1$ and so $D/D(w-\lambda) \cong D/D(w-1) \cong S_{def}$. Let S denote either $Skly_3$ or S_{def}, and let M be a simple finite-dimensional left S-module. Then we can construct from M a left D-module with D-action given by $d * m = \pi(d) \cdot m$. This module remains simple and finite-dimensional.

Conversely, given a simple finite-dimensional left D-module M, we will show that it belongs to either $\mathrm{Simp}_{<\infty}(Skly_3)$ or $\mathrm{Simp}_{<\infty}(S_{def})$. Since $k[w]$ embeds into D, we can consider ${}_{k[w]}M$, a finite-dimensional module over the PID, $k[w]$. Hence ${}_{k[w]}M$ is torsion and there exists a nonzero polynomial $f(w)$ in $\mathrm{ann}_{k[w]}M$. Now $k[w]/(f(w))$ embeds in $\mathrm{End}_D M$, a division ring by Schur's lemma. Thus $k[w]/(f(w))$ embeds into a domain, and $f(w)$ is irreducible and of the form $w - \lambda$ where $\lambda \in k$. If $\lambda = 0$, then $M \in \mathrm{Simp}_{<\infty}(Skly_3)$. Otherwise $D/D(w-\lambda) \cong D/D(w-1)$ by rescaling, and we have that M belongs to $\mathrm{Simp}_{<\infty}(S_{def})$. $\square$

Hence we proceed to study the simple finite-dimensional modules of $Skly_3$ in the next section. Results on the analysis of $\mathrm{Simp}_{<\infty}D$ is presented in Chapter 5.

4.3 Irreducible finite dimensional representations of Sklyanin algebras

We study irreducible finite-dimensional representations of the three-dimensional Sklyanin algebra (Definition IV.1), denoted by $Skly_3$, $S(a,b,c)$, or S. Before we begin, we point out that the representation theory of $Skly_3$ has also physical consequences.

Remark IV.8. We see specifically from Remark IV.5 that the algebra $S(a,b,c)$ arises as the superpotential algebra, $k\{x,y,z\}/(\partial\Phi_{marg})$. The corresponding marginal deformed theories are studied in [BJL00, §4.6.1], and in our language, the authors sought to understand the set $\mathrm{Simp}_{<\infty}S(a,b,c)$. This task was not achieved.

Recall from §2.1.3 that $Skly_3$ is associated to the geometric data $(E,\mathcal{L},\sigma)$ where

$$E = \mathbb{V}((abc)(x^3+y^3+z^3)-(a^3+b^3+c^3)(xyz)) \overset{i}{\subset} \mathbb{P}^2_k,$$

$\mathcal{L}$ is the invertible sheaf $i^*\mathcal{O}_{\mathbb{P}^2}(1)$ on E, and σ is the automorphism of E induced by the shift functor on point modules of S.

Assumption IV.9. We restrict our attention to the case that E is smooth curve, to say that the parameters a, b, c of S satisfy $abc \neq 0$ with $(3abc)^3 \neq (a^3+b^3+c^3)^3$. Furthermore we assume that $\sigma \in \mathrm{Aut}(E)$ is given by translation by a point so that our definition of $S(a,b,c)$ corresponds to the standard definition of a three-dimensional Sklyanin algebra, i.e. S is a type A Artin-Schelter regular algebra of dimension 3 [ATVdB90, 4.13].

Remark IV.10. Now the sheaf $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^2}(1)$ on E has degree 3 due to a version of the Riemann-Roch theorem. Namely if $\mathcal{L}$ is any line bundle on an irreducible, reduced projective curve C, then

$$\dim_k H^0(C,\mathcal{L}) - \dim_k H^1(C,\mathcal{L}) \;=\; \deg\mathcal{L} + 1 - \dim_k H^1(C,\mathcal{O}_C).$$

We also remind the reader that there exists a central, regular element g, homogeneous of degree 3 so that S/Sg is isomorphic to a twisted homogeneous coordinate ring $B = B(E,\mathcal{L},\sigma)$ (Theorem II.16).

The results of this section are given as follows. The simple 1-dimensional S-modules are determined in Lemma IV.11. We then consider the order of σ, and investigate $\mathrm{Simp}_{<\infty} S$ when $|\sigma| = \infty$ (Proposition IV.13), and when $|\sigma| < \infty$ afterward. For the $|\sigma| < \infty$ case in particular, we also take into consideration the subcases where simple finite-dimensional S-modules are either g-torsionfree or g-torsion. These results are reported in Propositions IV.16 and IV.18, respectively.

Recall the terminology from Definition II.31.

Lemma IV.11. *The set $Simp_1(S)$ solely consists of the trivial module: $\{S/S_+\}$.*

Proof. We know that:

$$\mathrm{Simp}_1(S) = \left\{ \begin{array}{ccc} S \xrightarrow{\phi} k & & a\beta\gamma + b\gamma\beta + c\alpha^2 = 0, \\ x \mapsto \alpha & & a\gamma\alpha + b\alpha\gamma + c\beta^2 = 0, \\ & : & \\ y \mapsto \beta & & a\alpha\beta + b\beta\alpha + c\gamma^2 = 0, \\ z \mapsto \gamma & & [a:b:c] \in \mathbb{P}^2_k \setminus \mathfrak{D} \end{array} \right\}.$$

By Assumption IV.9, $abc \neq 0$ so let $\lambda := -(a+b)/c$. Now the conditions on (α,β,γ) above are equivalent to:

$$\{\lambda\beta\gamma = \alpha^2,\ \lambda\alpha\gamma = \beta^2,\ \lambda\alpha\beta = \gamma^2\},$$

in which we have that $(\lambda^3 - 1)\alpha^2\beta = 0$. If $\lambda^3 \neq 1$, then we get that $\alpha = \beta = \gamma = 0$ and $\mathrm{Simp}_1(S) = \{S/S_+\}$. Else if $\lambda^3 = 1$, then $a^3 + b^3 + c^3 = -3ab(a+b)$. This implies that $(a^3 + b^3 + c^3)^3 = (3abc)^3$, which contradicts Assumption IV.9. $\square$

The next result describes $\mathrm{Simp}_{<\infty} S$ in the case that $|\sigma| = \infty$. First we require the following lemma.

Lemma IV.12. *Let A be a finitely generated, locally finite, connected graded k-algebra. Take $M \in Simp_{<\infty} A$ and let P be the largest graded ideal contained in $ann_A(M)$. Then we have that GKdim(A/P)= 0 or 1. In particular, GKdim(A/P)=0 if and only if M is the trivial module A/A_+.*

Proof. Let $\overline{A} \coloneqq A/P$. Note that $M \in \mathrm{Simp}_{<\infty}(\overline{A})$ so $\mathrm{ann}_{\overline{A}}(M)$ contains no homogeneous elements. Suppose by way of contradiction that $\mathrm{GKdim}(\overline{A}) > 1$. Then by Proposition II.27(5), we have that $\lim_{i\to\infty} \dim_k \overline{A}_i = \infty$. Since $\overline{A}/\mathrm{ann}_{\overline{A}}(M)$ is finite-dimensional by the Density Theorem [MR01, Theorem 0.3.6], we get that $\mathrm{ann}_{\overline{A}} M \cap \overline{A}_i \neq 0$ for all $i >> 0$. Hence there exists a homogeneous element in $\mathrm{ann}_{\overline{A}}(M)$, which contradicts the maximality of P. Now $\mathrm{GKdim}(\overline{A}) = 0$ or 1 by Proposition II.27(4).

If $M = A/A_+$, then $P = A_+$, and GKdim(A/P)=0 as A/P is finite-dimensional (Proposition II.27(3)). Conversely, GKdim(A/P) = 0 implies that $\overline{A} = A/P$ is finite-dimensional and connected graded. Since $\bigcap_{n\in\mathbb{N}} \left(\overline{A}_+\right)^n = 0$, we get that $\left(\overline{A}_+\right)^n = 0$ for some $n \in N$. Hence $\overline{A}_+ = 0$ by primality, and so $P = A_+$ with M being the trivial module. $\square$

Proposition IV.13. *The set $Simp_{<\infty}(S)$ equals $\{S/S_+\}$ when $|\sigma| = \infty$.*

Proof. According to Lemma II.32, we have that simple finite dimensional modules over S are quotients of irreducible objects in S-qgr. For $|\sigma| = \infty$, the set of nontrivial,

g-torsionfree quotients of S-qgr is empty due to [ATVdB91, Propositions 7.5, 7.9].

On the other hand, the set of g-torsion irreducible objects of S-qgr equals the set of irreducible objects of B-qgr where B is the twisted homogeneous coordinate ring $B(E, \mathcal{L}, \sigma)$ (Theorem II.16). This is precisely the set of B-point modules (Theorem II.14). Take $M \in \text{Simp}_{<\infty} B$ and let P be the largest graded ideal contained in $\text{ann}_B M$. By Lemma IV.12, we have that GKdim$(B/P) \leq 1$. If equal to 1, then Kdim(B/P)=1, which is a contradiction as $|\sigma| = \infty$ and B is projectively simple in this case [RRZ06]. Thus GKdim(B/P)=0, and again by Lemma IV.12 we know that $M = B/B_+$. □

We now determine the dimensions of simple finite-dimensional left S-modules in terms of the PI degree of S. We require the following preliminary result.

Notation. Let g denote the central homogeneous degree 3 element of S such that $S/Sg \cong B$ (Theorem II.16). Put $\Lambda := S[g^{-1}]$. We have that S is module finite over its center precisely when $|\sigma| = n < \infty$ (Theorem I.18). Thus S is PI and let p denote the PI degree of S in this case.

Note that the result below is mentioned in [LB95a, §3], yet the details are omitted. We provide the full proof here.

Lemma IV.14. *In the case of $|\sigma| < \infty$, the PIdeg(Λ_0) equals p/d where d =gcd$(3, |\sigma|)$.*

Proof. The center of S, denoted by $Z(S)$, has four generators: g of degree 3 and three others of degree $|\sigma|$ [ST94, Theorem 3.7]. Let u be any one of the degree $|\sigma|$ elements.

In the case of $d = 1$, recall that $\Lambda := S[g^{-1}]$ and consider the following notation:

$$\Lambda_0' := \Lambda_0[g^{\pm 1}], \quad \Lambda_0'' := \Lambda_0'[u^{\pm 1}], \quad \Lambda' := \Lambda[u^{\pm 1}].$$

Note that

$$\Lambda_0 \subseteq \Lambda_0' \subseteq \Lambda_0'' \subseteq \Lambda'.$$

Adjoining a central element does not alter PI degree so we have that:

(†) $\mathrm{PIdeg}(\Lambda_0) = \mathrm{PIdeg}(\Lambda_0') = \mathrm{PIdeg}(\Lambda_0'')$;

(‡) $\mathrm{PIdeg}(\Lambda') = \mathrm{PIdeg}(S) =: p$.

We now show that $\Lambda_0'' = \Lambda'$. Take $a \in \Lambda'$ of degree $m \in \mathbb{Z}$. Then there exists $m_1 \in \mathbb{Z}_{\geq 0}$ so that $au^{m_1} \in \Lambda$. As gcd(3,deg(u)) = 1, we have that $au^{m_1+m_2} \in \Lambda_{3\mathbb{Z}}$ for $m_2 = 0$, 1, or 2. Now $au^{m_1+m_2}g^{-t} \in \Lambda_0$ for some $t \in \mathbb{Z}_{\geq 0}$. Thus $a = (au^{m_1+m_2}g^{-t})(u^{-(m_1+m_2)}g^t) \in \Lambda_0''$.

Thus with (†,‡) above, we conclude that:

$$\mathrm{PIdeg}(\Lambda_0) = \mathrm{PIdeg}(\Lambda_0'') = \mathrm{PIdeg}(\Lambda') = p$$

as desired.

On the other hand, for $d = 3$ we show that:

$$\tfrac{p}{3} = \mathrm{PIdeg}(S_{3\mathbb{Z}}) = \mathrm{PIdeg}(\Lambda_0).$$

The second equality follows from two observations:

$$\begin{array}{lrl} (*) & \Lambda_0 & = \{sg^{-m} \ : \ m \in \mathbb{Z}, \ s \in S_{3m}\} = S_{3\mathbb{Z}}[g^{-1}]_0, \\ (**) & S_{3\mathbb{Z}}[g^{-1}] & = \{sg^{-m} \ : \ s \in S_{3d}; \ d, m \in \mathbb{Z}\} = S_{3\mathbb{Z}}[g^{-1}]_0[g^{\pm 1}]. \end{array}$$

Now we have that:

$$\mathrm{PIdeg}(S_{3\mathbb{Z}}) \overset{(**)}{=} \mathrm{PIdeg}(S_{3\mathbb{Z}}[g^{-1}]_0[g^{\pm 1}]) = \mathrm{PIdeg}(S_{3\mathbb{Z}}[g^{-1}]_0) \overset{(*)}{=} \mathrm{PIdeg}(\Lambda_0).$$

It suffices to show $p = 3 \cdot \mathrm{PIdeg}(S_{3\mathbb{Z}})$ on the level of fraction fields, i.e. we must show that $\mathrm{PIdeg}(FS) = 3(\mathrm{PIdeg}(FS_{3\mathbb{Z}}))$ for $F = \mathrm{Frac}(Z(S)) = Z(S)[C^{-1}]$ with $C = Z(S) \setminus \{0\}$. We also note that

$$F \ \subseteq \ FS_{3\mathbb{Z}} = S_{3\mathbb{Z}}[C^{-1}] \ \subseteq \ FS = S[C^{-1}].$$

Since S and $S_{3\mathbb{Z}}$ are domains, by Posner's theorem (Theorem II.42) we have that $FS_{3\mathbb{Z}}$ and FS are division rings. Let r denote the PIdegree of $FS_{3\mathbb{Z}}$. By Posner's theorem and [MR01, Lemma 13.3.4], we know that there exists a maximal subfield K of $FS_{3\mathbb{Z}}$ so that $\dim_K(FS_{3\mathbb{Z}}) = r$. Since $\dim_{S_{3\mathbb{Z}}} S = 3$, we have that $\dim_K(FS) = 3r$. Using the regular representation,

$$FS \hookrightarrow \mathrm{End}_K({}_K FS) \cong \mathrm{Mat}_{3r}(K),$$

we get that PIdeg$(FS) \leq 3r$.

For the other inequality, pick a maximal subfield L of FS containing K. Then

$$3r = \dim_K FS = (\dim_K L)(\dim_L FS) = (\dim_K L)(\mathrm{PIdeg} FS).$$

The last equality is again due to Posner's theorem and [MR01, Lemma 13.3.4]. Therefore, PIdeg(FS) divides $3r$.

Since the degree of central elements of $Z(S)$ are in $3\mathbb{Z}$, then

$$F = Z(S)[C^{-1}] \;\subseteq\; Z(FS_{3\mathbb{Z}}) = Z(S_{3\mathbb{Z}}[C^{-1}]).$$

Now we know that

$$(\dim_F FS_{3\mathbb{Z}})^{1/2} \;\geq\; (\dim_{Z(FS_{3\mathbb{Z}})} FS_{3\mathbb{Z}})^{1/2} \;=\; \mathrm{PIdeg} FS_{3\mathbb{Z}}.$$

Therefore

$$\begin{aligned} 3^{1/2}(\mathrm{PIdeg} FS_{3\mathbb{Z}}) &\leq (\dim_F FS_{3\mathbb{Z}})^{1/2} \cdot 3^{1/2} \\ &= (\dim_F FS_{3\mathbb{Z}})^{1/2} (\dim_{FS_{3\mathbb{Z}}} FS)^{1/2} \\ &= \mathrm{PIdeg} FS. \end{aligned}$$

In other words, $3^{1/2} \cdot r \leq \mathrm{PIdeg} FS$. Since PIdeg$FS$ divides $3r$ and PIdeg$FS > r$, we have that PIdeg$FS = 3r$. Thus

$$\mathrm{PIdeg} S \;=\; \mathrm{PIdeg} FS \;=\; 3 \cdot (\mathrm{PIdeg} FS_{3\mathbb{Z}}) \;=\; 3 \cdot (\mathrm{PIdeg} S_{3\mathbb{Z}}),$$

and PIdeg$S = 3 \cdot$PIdegΛ_0 in the case of $(3, |\sigma|) \neq 1$. $\square$

Corollary IV.15. *In this case $|\sigma| < \infty$, we have that the PI degree of $S(a,b,c)$ is equal to $|\sigma_{abc}|$.*

Proof. Let s denote the smallest integer such that σ^s fixes $[\mathcal{L}]$ in Pic$E$. We know by [ATVdB91, Theorem 7.3] that the ring Λ_0 is Azumaya (Definition II.46) and s is its PI degree. Now suppose that $(3, |\sigma|) = 1$. Then $s = |\sigma|$ by [Art92, §5]. Therefore with Lemma IV.14, we have that PIdeg S = PIdeg $\Lambda_0 = s = |\sigma|$.

On the other hand, suppose that $|\sigma|$ is divisible by 3. We have that s is also the order of the automorphism η introduced in [ATVdB91, §5], due to [ATVdB91, Theorem 7.3] (or more explicitly by [AdJ, Lemma 5.5.5(i)]). Since $\mathcal{L}$ has degree 3 (Remark IV.10), we know that η is σ^3 [AdJ, Lemma 5.3.6]. Thus

$$\text{PIdeg } S \;=\; 3 \cdot \text{PIdeg } \Lambda_0 \;=\; 3s \;=\; 3|\eta| \;=\; 3|\sigma^3| \;=\; 3 \cdot \frac{|\sigma|}{(3, |\sigma|)} \;=\; |\sigma|. \qquad \square$$

Now we consider two subcases of the classification of $\text{Simp}_{<\infty} S$ for $|\sigma| < \infty$: first consisting of g-torsionfree modules in Proposition IV.16, and secondly consisting of g-torsion modules in Proposition IV.18.

Proposition IV.16. *Take $|\sigma| < \infty$, and let M be a g-torsionfree simple finite-dimensional left S-module. Then we have that $\dim_k M = p$ for $(3, |\sigma|) = 1$, and $p/3 \le \dim_k M \le p$ for $|\sigma|$ divisible by 3.*

Proof. Let M be a g-torsionfree simple S-module of dimension $m < \infty$. By Lemma II.32 and Remark II.33, M is the quotient of some 1-critical graded S-module N with $\text{mult}(N) \le m$. Note that N is an irreducible object in S-qgr, which is also g-torsionfree. The equivalence of categories, $\underset{g-\text{torsionfree}}{\text{Irred}(S - \text{qgr})} \sim \text{Simp}_{<\infty}(\Lambda_0)$, from [ATVdB91, Theorem 7.5] implies that N yields the object $N[g^{-1}]_0$ in $\text{Simp}_{\infty}(\Lambda_0)$. Furthermore we have that $\dim_k N[g^{-1}]_0 = \text{mult}(N)$ as follows.

Since N is 1-critical, by Remark II.23 and Definition II.24 we know that $\dim_k(N_j) = \text{mult}(N)$ for $j >> 0$. Recall that g is homogeneous of degree 3. Note that $N[g^{-1}]_0 \cdot g^i \subseteq N_{3i}$ and moreover that $\dim_k(N_{3i}) = \text{mult}(N)$ for $i >> 0$. Hence such an i, we have that:

$$\dim_k N[g^{-1}]_0 \;=\; \dim_k(N[g^{-1}]_0 \cdot g^i) \;\leq\; \dim_k N_{3i} \;=\; \text{mult}(N).$$

Conversely $N_{3i} \cdot g^{-i} \subseteq N[g^{-1}]_0$. Hence for $i >> 0$ we have that:

$$\text{mult}(N) \;=\; \dim_k(N_{3i}) \;=\; \dim_k(N_{3i} \cdot g^{-i}) \;\leq\; \dim_k N[g^{-1}]_0.$$

Thus $\dim_k N[g^{-1}]_0 = \text{mult}(N)$ as desired.

Since Λ_0 is Azumaya, the Λ_0-module $N[g^{-1}]_0$ has dimension equal to $\text{PIdeg}(\Lambda_0)$ (Theorem II.51). Therefore $\text{PIdeg}(\Lambda_0) = \text{mult}(N) \leq m$. On the other hand, we know by Proposition II.52 that $m \leq p$. Hence

$$\text{PIdeg}(\Lambda_0) \;\leq\; \dim_k M \;\leq\; p.$$

In the case of $(3, |\sigma|) = 1$, Lemma IV.14 implies that $\text{PIdeg}\Lambda_0 = p$. Thus $\dim_k M = p$ in this case. For $|\sigma|$ divisible by 3, we can conclude from Lemma IV.14 that $p/3 \leq \dim_k M \leq p$. □

Corollary IV.17. *In the case of $|\sigma| < \infty$, a generic finite-dimensional simple, g-torsionfree, left S-module M has dimension equal to $|\sigma|$.*

Proof. We have an inclusion of sets $\underset{g-\text{torsionfree}}{\text{Simp}_{<\infty} S} \subset \text{Simp}_{<\infty} S[g^{-1}]$ given by $M \mapsto M[g^{-1}]$, where $M[g^{-1}]$ remains simple by [GW04, Corollary 10.16] and $\dim_k M[g^{-1}] = \dim_k M$ by [GW04, Exercise 10K]. Since inverting central elements does not alter PI degree, we have that $\text{PIdeg}(S[g^{-1}]) = \text{PIdeg}(S)$. Now we can conclude the result by applying Corollary IV.15 and Propositions II.52, II.54. □

To study g-torsion S-modules, let us consider the following notation.

Notation. Recall that S/Sg is isomorphic to a twisted homogeneous coordinate ring B (Theorem II.16). Since B is a homomorphic image of S which is PI for $|\sigma| < \infty$, we have that B is also PI in this case. Let q denote the PI degree of B.

Proposition IV.18. *For the case of $|\sigma| = n < \infty$, let M be a non-trivial g-torsion simple finite-dimensional left S-module. Then* $\dim_k M = q$.

Proof. For a graded ring $A = \bigoplus_{i \in \mathbb{N}} A_i$, let $\mathrm{Simp}^o_{<\infty} A$ denote the set of simple finite-dimensional left A-modules that are not annihilated by the irrelevant ideal $A_+ = \bigoplus_{i \geq 1} A_i$. Notice that

$$\underset{g-\text{torsion}}{\mathrm{Simp}_{<\infty} S} = \mathrm{Simp}_{<\infty} B.$$

Thus it suffices to show that all modules in $\mathrm{Simp}^o_{<\infty} B$ have maximal dimension, which is equal to q by Proposition II.52. We proceed by establishing the following claims.

Claim 1: We can reduce the task to studying $\mathrm{Simp}^o_{<\infty} \overline{B}$ for $\overline{B}$ some factor of B. Furthermore $\mathrm{Simp}^o_{<\infty} \overline{B} = \mathrm{Simp}^o_{<\infty} C$ for $C = B(Y_n, \mathcal{L}|_{Y_n}, \sigma|_{Y_n})$, the twisted homogeneous coordinate ring of an irreducible $|\sigma|$-orbit of E.

Claim 2: The ring C is isomorphic to a graded matrix ring.

Claim 3: The modules of $\mathrm{Simp}^o_{<\infty} C$ all have maximal dimension which is equal to $\mathrm{PIdeg}(C) = |\sigma|$.

Proof of Claim 1: Take $M \in \mathrm{Simp}^o_{<\infty} B$ and let P be the largest graded ideal contained in $\mathrm{ann}_B M$. Note that P is a prime ideal as follows. If not, then there exists graded ideals I, J strictly containing P, with $IJ \subseteq P$. Now $IJM = 0$, yet $JM \neq 0$. Hence by the simplicity of M, we know that $JM = M$. So $IM = 0$, which prompts a contradiction. Set $\overline{B} = B/P$.

We have by Lemma IV.12 that GKdim($\overline{B}$)=1, so Kdim($\overline{B}$) = 1 [MR01, 8.3.18]. We also have by [AS95, Lemma 4.4] that the height one prime P of B corresponds to an σ-irreducible maximal closed subset of E. Namely as σ is given by translation, P is associated to a σ-orbit of $|\sigma|$ points of E, an orbit denoted by Y_n. Here $n := |\sigma|$. Now by [BRS10, proof of Proposition 3.8(1)], we have a natural homomorphism:

$$\phi : B \longrightarrow B(Y_n, \mathcal{L}|_{Y_n}, \sigma|_{Y_n}) =: C$$

given by restrictions of sections, with $\ker(\phi) = P$. The map ϕ is also surjective in high degree, whence $\overline{B}$ is isomorphic to C in high degree.

Since $\overline{B}$ and C are PI, we have by Kaplansky's theorem [MR01, Theorem 13.3.8] that for each of these rings: the maximal spectrum equals the primitive spectrum. Thus it suffices to verify the following subclaim.

<u>Subclaim:</u> For a graded k-algebra $A = \bigoplus_{i\geq 0} A_i$, let $\max^o A$ denote the set of maximal ideals of A not containing the irrelevant ideal A_+. Then there is a bijective correspondence between $\max^o C$ and $\max^o \overline{B}$, given by $I \mapsto \overline{B} \cap I$. Moreover $\overline{B}/(\overline{B} \cap I) \cong C/I$.

Proof of Subclaim: We know that $\overline{B}_{\geq m} = C_{\geq m}$ for $m >> 0$. Therefore as B is generated in degree 1, for any such m:

$$J := (\overline{B}_+)^m = \overline{B}_{\geq m} = C_{\geq m} \supseteq (C_+)^m,$$

is a common ideal of $\overline{B}$ and C. (The last inclusion may be strict as C need not be generated in degree 1.)

For one inclusion, let $I \in \max^o C$. We want to show that $\overline{B} \cap I \in \max^o \overline{B}$. Note that $\overline{B}/(\overline{B} \cap I) \cong (\overline{B} + I)/I$ as rings. Furthermore

$$C \;\supseteq\; \overline{B} + I \;\supseteq\; J + I \;\supseteq\; (C_+)^m + I = C.$$

The last equality is due to C_+ and I being comaximal in C. Hence $C = \overline{B} + I$ and $\overline{B}/(\overline{B} \cap I) \cong C/I$ is a simple ring. Consequently $\overline{B} \cap I \in \max^o \overline{B}$.

Conversely, take $M \in \max^o \overline{B}$ and we want to show that $M = \overline{B} \cap Q$ for some $Q \in \max^o C$. We show that $CMC \neq C$. Suppose not, then

$$(\overline{B}_+)^{2m} = J^2 = JCJ = JCMCJ = JMJ \subseteq M,$$

which implies that $\overline{B}_+ \subseteq M$ as M is prime. This is a contradiction. Now CMC is contained in some maximal ideal Q of C. Since $M \in \max^o \overline{B}$, we have that $Q \in \max^o C$; else $M \subseteq Q \cap \overline{B} = \overline{B}_+$. By the last paragraph, we know that $\overline{B} \cap Q \in \max^o \overline{B}$ with $\overline{B} \cap Q = M \in \max \overline{B}$.

Thus Claim 1 is verified and so it suffices to examine $\mathrm{Simp}^o_{<\infty} C$. $\square$

Proof of Claim 2: Recall that $n := |\sigma|$. We will show that C is isomorphic to the graded matrix ring:

$$R := \begin{pmatrix} T & x^{n-1}T & x^{n-2}T & \dots & xT \\ xT & T & x^{n-1}T & \dots & x^2T \\ x^2T & xT & T & \dots & x^3T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{n-1}T & x^{n-2}T & x^{n-3}T & \dots & T \end{pmatrix}$$

with $T = k[x^n]$. Say $Y_n := \{p_1, \dots, p_n\}$, the $|\sigma|$-orbit of n distinct points p_i. Let $\mathcal{L}' := \mathcal{L}|_{Y_n}$ and $\sigma' := \sigma|_{Y_n}$. Reorder the $\{p_i\}$ to assume that $\sigma'(p_i) = p_{i+1}$ and $\sigma'(p_n) = p_1$ for $1 \leq i \leq n-1$. Now by Definition II.9, $C = \bigoplus_{d \geq 0} H^0(Y_n, \mathcal{L}'_d)$ where $\mathcal{L}'_0 = \bigoplus_{i=1}^n \mathcal{O}_{p_i}$, and $\mathcal{L}'_1 = \bigoplus_{i=1}^n \mathcal{O}_{p_i}(1)$, and

$$\mathcal{L}'_d = \mathcal{L}' \otimes_{\mathcal{O}_{Y_n}} (\mathcal{L}')^{\sigma'} \otimes_{\mathcal{O}_{Y_n}} \dots \otimes_{\mathcal{O}_{Y_n}} (\mathcal{L}')^{{\sigma'}^{d-1}}.$$

Note that

$$(\mathcal{L}')^{\sigma} = \mathcal{O}_{p_n}(1) \oplus \mathcal{O}_{p_1}(1) \oplus \dots \oplus \mathcal{O}_{p_{n-1}}(1) \cong \mathcal{L}'.$$

Therefore $\mathcal{L}'_d \cong (\mathcal{L}')^{\otimes d}$. If $i = j$, then $\mathcal{O}_{p_i}(1) \otimes \mathcal{O}_{p_j}(1) \cong \mathcal{O}_{p_i}(2)$. Otherwise the sheaf $\mathcal{O}_{p_i}(1) \otimes \mathcal{O}_{p_j}(1)$ has empty support. Hence $(\mathcal{L}')^{\otimes d} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{p_i}(d)$. So C shares the same k-vector space structure as a sum of n polynomial rings in one variable, say $k[u_1] \oplus \cdots \oplus k[u_n]$.

We next study the multiplication of C; write C_i as $k \cdot u_1^i \oplus \cdots \oplus k \cdot u_n^i$. We define the multiplication $C_i \times C_j \to C_{i+j}$ on basis elements $(u_1^k, \ldots, u_n^k)$ for $k = i, j$, which will extend to C by linearity. Moreover all of the following sums in indices are taken modulo n.

Observe that

$$\begin{aligned} C_i \times C_j &= H^0(Y_n, \mathcal{L}'_i) \otimes H^0(Y_n, \mathcal{L}'_j) \\ &= H^0(Y_n, \mathcal{L}'_i) \otimes H^0(Y_n, (\mathcal{L}'_j)^{(\sigma')^i}). \end{aligned}$$

Recall the multiplicative structure of twisted homogeneous coordinate rings (Definition II.9). Hence the multiplication C is defined as

$$(u_1^i, \ldots, u_n^i) * (u_1^j, \ldots, u_n^j) = (u_1^i u_{1-i}^j, \ldots, u_n^i u_{n-i}^j).$$

We now show that C is isomorphic to the graded matrix ring R. Define a map $\phi : C \to R$ by

$$(u_1^i, \ldots, u_n^i) \mapsto \begin{pmatrix} \mathrm{Row}(u_1^i) \\ \vdots \\ \mathrm{Row}(u_n^i) \end{pmatrix} =: \mathtt{M_i}.$$

Here $\mathrm{Row}(u_l^i)$ denotes the row with the entry u_l^i in column $l-i$ modulo n, and zeros entries elsewhere. In other words, the matrix $\mathtt{M_i}$ has a degree i entry in positions $(l, l-i)$ for $l = 1, \ldots, n$ and zeros elsewhere. We see that $\mathtt{M_i} \in R$.

The map ϕ is the ring homomorphism as follows. First note that

$$\phi((u_1^i,\ldots,u_n^i) * (u_1^j,\ldots,u_n^j)) = \phi((u_1^i u_{1-i}^j,\ldots,u_n^i u_{n-i}^j)) = \begin{pmatrix} \mathrm{Row}(u_1^i u_{1-i}^j) \\ \vdots \\ \mathrm{Row}(u_n^i u_{n-i}^j) \end{pmatrix}.$$

Here the entry $u_l^i u_{l-i}^j$ appears in positions $(l, l-(i+j))$ for $l = 1,\ldots,n$.

On the other hand,

$$\phi((u_1^i,\ldots,u_n^i)) \cdot \phi((u_1^j,\ldots,u_n^j)) = \begin{pmatrix} \mathrm{Row}(u_1^i) \\ \vdots \\ \mathrm{Row}(u_n^i) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Row}(u_1^j) \\ \vdots \\ \mathrm{Row}(u_n^j) \end{pmatrix} =: \mathtt{M_i} \cdot \mathtt{M_j}.$$

The nonzero entries of $\mathtt{M_i}$, namely u_l^i, appear in positions $(l, l-i)$ for $l = 1,\ldots,n$, whereas the nonzero entries of $\mathtt{M_j}$, the u_l^j, are in positions $(l, l-j) = (l-i, l-(i+j))$ for $l, i = 1,\ldots,n$. Therefore the product $\mathtt{M_i} \cdot \mathtt{M_j}$ has nonzero entries in positions $(l, l-(i+j))$ and these entries are $u_l^i u_{l-i}^j$ for $l = 1,\ldots,n$. Thus we have a ring homomorphism ϕ between C and R which is clearly bijective. This conclude the proof of Claim 2.

$\square$

Proof of Claim 3: We aim to show that $M \in \mathrm{Simp}^o_{<\infty} R$ has maximal k-vector space dimension, which is equal to n. Let s denote the diagonal matrix, $\mathrm{diag}(x^n)$ in R, and note that $sR = R_+$. Consider the ring $R[s^{-1}]$. For $M \in \mathrm{Simp}^o_{<\infty} R$, construct the module $0 \neq M[s^{-1}] = R[s^{-1}] \otimes_R M \in R[s^{-1}]$-mod. Since M is simple, $M[s^{-1}]$ is also simple by [GW04, Corollary 10.16]. Furthermore $\dim_k M[s^{-1}] = \dim_k M$ as M is s-torsionfree [GW04, Exercise 10K]. Thus we have an inclusion of sets $\mathrm{Simp}^o_{<\infty} R \subseteq$

$\mathrm{Simp}_{<\infty}R[s^{-1}]$ given by $M \mapsto M[s^{-1}]$. Now observe that:

$$R[s^{-1}] = \begin{pmatrix} T' & x^{n-1}T' & & xT' \\ xT' & T' & & x^2T' \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1}T' & x^{n-2}T' & & T' \end{pmatrix}$$

for $T' = k[(x^n)^{\pm 1}]$. So $R[s^{-1}] \subseteq \mathrm{Mat}_n(T')$ as rings, to say the rings have the same multiplicative structure. The rings are also equal as sets. Hence $R[s^{-1}] = \mathrm{Mat}_n(T')$ as rings. Since all simple modules of $\mathrm{Mat}_n(T')$ have dimension n, we have now verified Claim 3. $\square$

Thus we conclude the proof of the proposition. $\square$

Corollary IV.19. *Given a three-dimensional Sklyanin algebra $S(a,b,c)$ where $|\sigma_{abc}| < \infty$ with $S/Sg \cong B$, we have that PIdeg(S) = PIdeg(B) = $|\sigma|$.*

Proof. In the proof of the preceding proposition, we have that $|\sigma|$ is equal to the maximal dimension of the simple finite-dimensional modules of the rings R and C. Since simple finite-dimensional modules of R and C correspond to modules in $\mathrm{Simp}^o_{<\infty}B$ with the same dimension by Claim 1, we have that $|\sigma| = \mathrm{PIdeg}(B)$ (Proposition II.52). Moreover $\mathrm{PIdeg}(S) = |\sigma|$ by Corollary IV.15. $\square$

Corollary IV.20. *(We remind the reader that we have restricted our attention to E smooth.) The smooth locus and Azumaya locus of B coincide when $|\sigma| < \infty$.*

Proof. By [ST94, Corollary 2.8], the center of B is equal to the twisted homogeneous coordinate ring associated to the triple $(E', \mathcal{L}|_{E'}, \sigma|_{E'})$ where $E' = \frac{E}{<\sigma>}$, which is also a smooth elliptic curve. Hence $\max Z(B)$ is the affine cone over $E' \subseteq \mathbb{P}^2$, and so the smooth locus of $\max Z(B)$ is equal to $\max Z(B) \setminus \{0\}$. This set is precisely the Azumaya locus as proved in Proposition IV.18. $\square$

CHAPTER V

Representation theory of deformed Sklyanin algebras II: On the central extension D of $Skly_3$

Recall from Chapter 4 that our main objective is to classify irreducible finite-dimensional representations of the deformed Sklyanin algebra S_{def} (Definition IV.2), a task which has implications in string theory (see §4.1). This problem boils down to studying such modules over the three-dimensional Sklyanin algebra $Skly_3$ (Definition IV.1), and over the central extension D of $Skly_3$. We are only able to give partial results towards this classification, and these are collected in this chapter. For instance, we are able to determine the 1-dimensional representations of D (Lemmas V.5 and V.6). For the reader's convenience, we restate the definition of D below.

Definition V.1. For $i = 1, 2, 3$, let a, b, c, d_i, e_i be scalars in k with $[a : b : c] \notin \mathfrak{D}$ (Definition IV.1). The **central extension** D **of** $Skly_3$ is generated by three variables x, y, z and a central element w, subject to the relations:

$$ayz + bzy + cx^2 + d_1xw + e_1w^2 = 0$$

$$azx + bxz + cy^2 + d_2yw + e_2w^2 = 0$$

$$axy + byx + cz^2 + d_3zw + e_3w^2 = 0$$

$$xw - wx \;=\; yw - wy \;=\; zw - wz \;=\; 0.$$

Notation. Let $\mathrm{Simp}_{<\infty}A$ denote the set of isomorphism classes of simple finite-dimensional left A-modules, and let $\mathrm{Simp}_m A$ denote those of dimension m.

Now the set $\mathrm{Simp}_{<\infty} Skly_3$ was investigated in Chapter 4 and hence this chapter is dedicated to the study of $\mathrm{Simp}_{<\infty} D$.

In the first section, we present results on the noncommutative geometry of D and we also describe the set $\mathrm{Simp}_1 D$ in terms of D-point modules. In certain cases (of physical significance), we provide the explicit descriptions of $\mathrm{Simp}_1 D$ and $\mathrm{Simp}_1 S_{def}$. The last two sections discuss partial results in determining the center of D and the fat point modules over D, with a view towards classifying higher dimensional irreducible representations of D. These techniques are motivated by the work of [LB95b, SS93] and their analysis of conformal $\mathfrak{sl}_2$ enveloping algebras and four-dimensional Sklyanin algebras respectively.

5.1 Background on the geometry of D, and computation of $\mathrm{Simp}_1 D$

Fortunately much is known about the noncommutative geometry of D, to say its point scheme and line scheme (Definition II.19) are known, and we present these results below. As a consequence, we can describe the 1-dimensional representations of D in terms of D-point modules.

To describe the geometric data of D, we remind the reader of the constructions of a point scheme with automorphism σ induced by the shift functor and of a line scheme (Definitions II.19, II.8). Recall in particular that the point scheme of $Skly_3$ is generically an elliptic curve $E \subseteq \mathbb{P}^2$ (Proposition II.15). In fact, we make the following assumption.

Assumption V.2. Let us restrict our attention to the case that the point scheme of $Skly_3$ is smooth, i.e. $E = \mathbb{P}^2$ or a smooth cubic curve. In the case that E is a smooth curve, we invoke Assumption IV.9 from Chapter 4.

Proposition V.3. *[LBSVdB96, Theorem 4.1.11, Theorem 4.2.2(2), Proposition*

4.3.7] Let D be a central extension of a generic three-dimensional Sklyanin algebra as in Definition V.1.

1. *Point modules $M(p)$ of D corresponding to $p \in P_D$ are the form of $D/Dy_1 + Dy_2 + Dy_3$ where $y_i \in D_1$. We also have that the point $p \in P_D$ is $\mathbb{V}(y_1, y_2, y_3) \subseteq \mathbb{P}^3$.*

2. *The point scheme of D, denoted P_D, is the union of E and a set of r points $\mathcal{S} = \{s_i\}_{i=1}^r$. In fact, $r = 8$ for generic central extensions D of $Skly_3$ and this is the maximum cardinality of $\mathcal{S}$. Here $E \subseteq \mathbb{V}(w) = \mathbb{P}^2 \subseteq \mathbb{P}^3_{[x:y:z:w]}$ and $\mathcal{S} \subseteq \mathbb{P}^3 \smallsetminus \mathbb{V}(w)$.*

3. *The automorphism σ_D of P_D, induced by the shift functor on point modules of D, is given by σ on E and the identity on $\mathcal{S}$.*

Observe that by the last part of this result, we get that $|\sigma_D| = |\sigma|$.

Proposition V.4. *[LBSVdB96, Example 5.2.5] The line scheme of D lies in $\mathbb{P}^3_{[x:y:z:w]}$ and is the union of the line scheme of $Skly_3$ and two lines passing through p for each point $p \in E \subseteq \mathbb{V}(w) \subseteq \mathbb{P}^2$.*

We will use the line scheme later to understand fat point modules of D. On the other hand, since we understand point modules over D, we can now study the 1-dimensional representations of D and those of S_{def}.

Lemma V.5. *Let D be a central extension of a three-dimensional Sklyanin algebra. Then the simple 1-dimensional D-modules arise as simple quotients of point modules over D.*

Proof. By Lemma II.32, a simple 1-dimensional D-module M is the quotient of some 1-critical graded D-module N. Moreover by Remark II.33, $\mathrm{mult}(N) \leq \dim_k M$=1, so $\mathrm{mult}(N)$=1. Thus simple 1-dimensional D-modules are quotients of D-point modules. □

Before we continue to study the sets $\mathrm{Simp}_1 D$ and $\mathrm{Simp}_1 S_{def}$, we point out specific algebras, D and S_{def}, that are of physical importance, namely relevant to [BJL00, §4]. The authors in this work study *q-deformed theories* or rather the set $\mathrm{Simp}_{<\infty} S_{def}(a,b,c,d_i,e_i)$ with the parameter $c = 0$. They also focus on the theories equipped with either: (i) a single *mass term*; (ii) a mass term and a *linear term*; (iii) three arbitrary linear terms; or (iv) three arbitrary mass terms. In our language, this is equivalent to classifying $\mathrm{Simp}_{<\infty} S_{def}$ with the respective conditions that:

(i) d_1 is arbitrary, $d_2 = d_3 = e_i = 0$;

(ii) d_1, e_1 are arbitrary, $d_2 = d_3 = e_2 = e_3 = 0$;

(iii) $d_i = 0$, or;

(iv) $e_i = 0$.

Thus we describe the 1-dimensional representations of S_{def} for each of the cases: $d_i = 0$, $e_i = 0$, and $d_2 = d_3 = e_2 = e_3 = 0$.

Lemma V.6. *For each of the cases $d_i = 0$, $e_i = 0$, and $d_2 = d_3 = e_2 = e_3 = 0$, the 1-dimensional representations of $S_{def}(a,b,c,d_i,e_i)$ can be explicitly described. In fact, there is one such representation if $d_i = 0$, five representations if $e_i = 0$, and three representations if $d_2 = d_3 = e_2 = e_3 = 0$.*

Proof. See Routine A.4. □

Now we compare these 1-dimensional representations of S_{def} with the following results on point modules of D.

Proposition V.7. *Let D be a central extension of a generic Sklyanin algebra. Then the non-trivial simple 1-dimensional modules over S_{def} are simple quotients of the point modules*

$$\{M(s_i) \mid s_i \in \mathcal{S}\} = \left\{ \frac{D}{Dy_1 + Dy_2 + Dy_3} \mid \mathbb{V}(y_1, y_2, y_3) = s_i \in P_D \right\}.$$

The set $\mathcal{S} = \{s_i\}$ is explicitly described in the appendix for the cases: $d_i = 0$, $e_i = 0$, and $d_2 = d_3 = e_2 = e_3 = 0$. In fact for these cases, $\mathcal{S}$ consists of 1 point, 5 points, and 3 points respectively.

Proof. By Lemma IV.7, we know that $\mathrm{Simp}_1 S_{def} = \mathrm{Simp}_1 D \smallsetminus \mathrm{Simp}_1 Skly_3$. Hence by Lemma IV.11, we have that $\mathrm{Simp}_1 S_{def} = \mathrm{Simp}_1 D \smallsetminus \{D/D_+\}$.

For generic D, we know that $|\sigma_D| = |\sigma| = \infty$, and so the point modules $M(p)$ for $p \in E$ only yield trivial simple quotients [LS93, Lemma 5.8(d)]. Thus the nontrivial simple 1-dimensional S_{def}-modules are simple quotients of the point modules $M(s_i)$ for $s_i \in \mathcal{S}$.

Given the parameters $d_i = 0$, $e_i = 0$, or $d_2 = d_3 = e_2 = e_3 = 0$, refer to Routine A.5 for the computation of (the point scheme P_D, and particularly) the set $\mathcal{S}$. □

5.2 On the computation of Simp$_{>1}D$

Since we studied 1-dimensional representations of D (and of S_{def}) in the previous section, we now analyze simple D-modules of higher dimension. We are not able to provide classification results in general, but we proceed by listing partial results towards this goal. We begin with the following result.

Lemma V.8. *If $M \in Simp_{<\infty} D$ arises as a simple quotient of a fat point module, then* $\dim_k M > 1$.

Proof. Apply Remark II.33. □

In fact for generic D, we would like to establish the nonexistence of D-fat point modules.

On the other hand for certain D, namely some for which $|\sigma_D| = |\sigma| < \infty$, we aim to verify that D is module-finite over its center. In this case, D would have many higher dimensional representations, particularly many of which have dimension equal to the PI degree of D. This is due to the Azumaya locus of D (Definition II.53) being dense in $\max Z(D)$ (Propositions II.52, II.54). This leads us to the following claim.

Conjecture V.9. *A central extension D of $Skly_3$ is either PI or all simple finite-dimensional D-modules are 1-dimensional.*

In §5.3, we discuss results on the center of D. We show that for generic D (for $|\sigma| = \infty$), the center of D is generated by the central element w and a homogeneous degree 3 element $\hat{g}_3$ (Proposition V.10). Moreover D is not PI in this case. We also investigate the center of D when $|\sigma| < \infty$, in which we conjectured that D is PI if $|\sigma| = 3, 6$ and not PI if $|\sigma| = 1, 2$ (Conjectures V.12, V.13, V.15, Remark V.16).

The final section, §5.4, is dedicated to the study of fat point modules of D for the reasons mentioned above. We aim to present fat point modules of D as quotients of line modules of D by shifted line modules of D, a strategy prompted by the methods in [LB95b] and [SS93]. We show that this presentation of D-fat point modules holds in the case that $|\sigma| = 2$ (Proposition V.29). We then aim to employ the geometry of line modules to draw conclusions about either the existence or structure of fat point modules. Partial results are reported below.

5.3 On the center of D

In this section we discuss the center of D and whether D satisfies a polynomial identity. Recall from the definition of D that D has one central element w of degree 1. For generic D (with $|\sigma_D| = |\sigma| = \infty$), we also have another central element $\hat{g}_3$ homogeneous of degree 3. This element is computed with the noncommutative algebra package Affine of the computer algebra system Maxima, and it is also presented in [EG, §9]. Here we assume (by rescaling) that the parameter a of D is equal to 1:

(5.1)

$$
\begin{aligned}
\hat{g}_3 = \ & c(c^3-b^3)y^3 + b(c^3-1)yxz + (b^3-c^3)xyz + c(1-c^3)x^3 \\
& +(-c^2d_1)wyz \\
& +[(-b^4d_2+b^3d_2-b^2d_2+2bc^3d_2-c^3d_2)/(b-1)]wy^2 \\
& +[(-b^3d_3+bc^3d_3)/(-c(b-1))]wyx \\
& +(bc^2d_2)wxz \\
& +[(-b^2d_3+bc^3d_3)/(-c(b-1))]wxy \\
& +[(-b^2d_1-bc^3d_1+bd_1+2c^3d_1-d_1)/(b-1)]wx^2 \\
& +[(-b^3ce_3+2b^2ce_3-b^2d_3^2+bc^2d_1d_2-bce_3/(-c(b-1))]w^2z \\
& +[(-b^4e_2+2b^3e_2-2b^2e_2+bc^3e_2+bc^2d_2^2-bcd_1d_3+be_2-c^3e_2)/(b-1)]w^2y \\
& +[(b^3e_1+2b^2e_1-bc^3e_1-bcd_2d_3+2be_1+c^3e_1+c^2d_1^2-e_1)/(b-1)]w^2x.
\end{aligned}
$$

Considering the surjection $D \twoheadrightarrow D/Dw \cong Skly_3$, we have that the image of $\hat{g}_3$ is the element g from Remark II.17. With this observation, we now show that $\hat{g}_3$ and w generate the center of D in the case where $|\sigma| = \infty$.

Proposition V.10. *Given a central extension D of $Skly_3$ for which $|\sigma| = \infty$, we have that $Z(D) = k[w, \hat{g}_3]$. In this case, D is not PI.*

Proof. Let C be the graded subalgebra of $Z(D)$ generated by $\hat{g}_3$. Consider the surjection $\pi : D \twoheadrightarrow D/Dw = Skly_3$. Since $\pi(C) = k[g]$, and $Z(S) = k[g]$ for $|\sigma| = \infty$ [Smi94, §13], we have by [ST94, Lemma 3.6] that $Z(D) = k[\hat{g}_3, w]$.

Furthermore S is not PI as $|\sigma| = \infty$ (Theorem I.18). So by considering the homomorphism π above, we have that D is also not PI (Proposition II.38). □

Hence according to Conjecture V.9, we claim that $\mathrm{Simp}_{<\infty}D = \mathrm{Simp}_1 D$ for $|\sigma| = \infty$. This problem is addressed in the next section.

Now we study the center of D when $|\sigma_D| = |\sigma| < \infty$. The task of finding such D is equivalent to classifying parameters (a,b,c) for which $\sigma = \sigma_{abc}$ has finite order. Here, by rescaling parameters, we assume that $a = 1$. Partial results on this problem are reported in Proposition A.1 in the appendix. In fact for $n = 1,\dots,6$, we know the parameters (a,b,c) for which $\sigma = \sigma_{abc}$ has order n. Hence we study the central extensions D of $S(a,b,c)$ where $|\sigma_{abc}| = 1,\dots,6$.

Case: $|\sigma| = 1$

First, we have that $|\sigma| = 1$ if and only if $Skly_3$ is the commutative polynomial ring $S(1,-1,0) = k[x,y,z]$. Contrary to the behavior of $k[x,y,z]$, for (generic) parameters d_i, e_i, we have the following computations for the center of a central extension of $k[x,y,z]$.

Lemma V.11. *1. Let D be a generic central extension of $k[x,y,z]$. Then for $1 \le n \le 3$, the k-vector spaces $Z(D)_n$ are generated by w and*

$$q = d_1x^2 + d_2y^2 + d_3z^2 + 2e_1wx + 2e_2wy + 2e_3wz.$$

2. Let D be the central extension of a $k[x,y,z]$ given in the table below. Then for $1 \le n \le 6$, the k-vector spaces $Z(D)_n$ are generated by w and the element q given as follows:

D	q
$D(1,-1,0,0,0,0,e_1,e_2,e_3)$	$e_1x + e_2y + e_3z$
$D(1,-1,0,d_1,d_2,d_3,0,0,0)$	$d_1x^2 + d_2y^2 + d_3z^2$
$D(1,-1,0,d_1,0,0,e_1,0,0)$	x

□

This leads us to the following claim.

Conjecture V.12. *The center of $D(1,-1,0,d_i,e_i)$ (respectively of D with $d_i = 0$, with $e_i = 0$, and with $d_2 = d_3 = e_2 = e_3 = 0$) is generated by the elements w and q given in Lemma V.11. Thus D is not PI.*

If this conjecture holds, then we see that a result such as Theorem I.18 clearly does not apply. In other words, the structure of $Z(D)$ does not necessarily depend on the order of σ_D $(= |\sigma|)$. However we will see that this relationship does hold for other cases, such as $|\sigma| = 3$.

We proceed to study the center of D with σ of higher order. Since the algebra program Affine does not accommodate for complex parameters and parameters with fractional powers, according to Proposition A.1 we restrict our attention to $|\sigma| = 2, 3, 6$.

Case: $|\sigma| = 2$

Here we have that $|\sigma| = 2$ if and only if the parameters (a,b,c) equal $(1,1,c)$ with $c \neq 0$ and $c^3 \neq 1$ by Proposition A.1. Moreover by Affine, we have a central element $\hat{g}_2$ of D of degree 2:

$$\begin{aligned} \hat{g}_2 &= d_2(c^4-c)y^2 + d_3(-c^3+1)yx + d_3(-c^3+1)xy + d_1(c^4-c)x^2 \\ &\quad + (d_3^2 - c^2 d_1 d_2)\cdot wz + (-c^2 d_1 d_3 + c^3 d_2^2)\cdot wy + (-c^2 d_2 d_3 + c^3 d_1^2)\cdot wx. \end{aligned} \tag{5.2}$$

Now for various parameters c, d_i, e_i, we have computed the center of D up to degree 12. For these examples, the k-vector spaces $Z(D)_n$ for $1 \leq n \leq 12$, are generated by w and $\hat{g}_2$. Hence we make the following claim.

Conjecture V.13. *For generic parameters d_i and e_i, the center of the central extension $D(1,1,c,d_i,e_i)$ is $k[\hat{g}_2, w]$. Moreover D is not PI.*

Case: $|\sigma| = 3$

By Proposition A.1, we know that $|\sigma_{abc}| = 3$ if and only if:

$$[a:b:c] = \begin{cases} [1:0:\omega], \ for \ \omega = -1, e^{\pi i/3}, e^{5\pi i/3}; \\ [1:\omega:0], \ for \ \omega = e^{\pi i/3}, e^{5\pi i/3}. \end{cases}.$$

For computational reasons mentioned above, we will focus on the case where $(a,b,c) = (1,0,-1)$.

Lemma V.14. *The central extension $D(1,0,-1,d_1,d_2,d_3,e_1,e_2,e_3)$ is module-finite over its center, and thus is PI. Furthermore $Z(D)$ is generated by four elements, $\{\rho_i\}_{i=1}^4$, of degree 3, and one element w of degree 1.*

Proof. Note that $S(1,0,-1) \cong D(1,0,-1,d_1,d_2,d_3,e_1,e_2,e_3)/(w)$ is module-finite over its center as $|\sigma_{1,0,-1}| = 3$ (Theorem I.18). Following [ST94, Lemma 3.6], we show that there exists a graded subalgebra C of $Z(D)$ so that the image of C in $S = S(1,0,-1)$ is $Z(S)$. By [ST94, Theorem 3.7], $Z(S)$ is generated by four elements of degree 3. In fact by Affine, we know that these generators are:

$$\begin{aligned} \rho_1 &= zyx + yxz + xzy, \\ \rho_2 &= y^2x + xy^2 + x^2z, \\ \rho_3 &= yx^2 + xyx + x^2y, \\ \rho_4 &= x^3. \end{aligned}$$

Furthermore, we have by Affine, four degree 3 central elements of D:

$$\begin{aligned} \hat{\rho_1} = \ & \rho_1 - d_1(wzy) - d_2(wy^2) - d_3(wyx) - d_2(wxz) - d_3(wxy) - d_1(wx^2) \\ & +(e_3 + d_1d_2)(w^2z) + (e_2 + d_1d_3 + d_2^2)(w^2y) + (e_1 + d_2d_3 + d_1^2)(w^2x), \\ \hat{\rho_2} = \ & \rho_2 - d_1(wy^2) - d_2(wyx) - d_1(wxz) - d_2(wxy) - d_3(wx^2) - e_1(w^2z) \\ & +d_1d_2(w^2y) + (-e_2 + d_1d_3)(w^2x), \\ \hat{\rho_3} = \ & \rho_3 - d_1(wyx) - d_1(wxy) - d_2(wx^2) - e_1(w^2y) + (e_3 + d_1d_2)(w^2x), \\ \hat{\rho_4} = \ & \rho_4 - d_1(wx^2) - e_1(w^2x). \end{aligned}$$

Let C be the graded subalgebra of $Z(D)$ generated by the $\hat{\rho}_i$. Since the image of $\hat{\rho}_i$ in S is ρ_i, we have by [ST94, Lemma 3.6] that both $Z(D) = k[\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4, w]$ and D is module-finite over this center. $\square$

Thus Conjecture V.9 holds for a central extension D with $|\sigma| = 3$. For the other four central extensions D where $|\sigma| = 3$, we believe that $Z(D)$ is generated by w, and elements $\{\hat{\rho}_{i,\omega}\}_{i=1}^{4}$ which are the elements $\hat{\rho}_i$ with coefficients involving ω and ω^2. This prompts the following claim.

Conjecture V.15. *Let D be a central extension of $Skly_3$ for which $|\sigma|$=3. Then the center of D is generated by w and four elements of degree 3, and D is module-finite over its center. In other words, Conjecture V.9 holds for $|\sigma|$=3.*

Cases: $|\sigma| = 4, 5$

If the automorphism σ_{abc} has order 4 or 5, then one of the parameters, b or c, is a complex number or involves a fractional power (Proposition A.1). For the computational reasons mentioned above, we will omit these cases for now.

Case: $|\sigma| = 6$

By Proposition A.1, we know that $|\sigma_{abc}| = 6$ if and only if:

$$[a:b:c] = \begin{cases} [1:b:\zeta], [1:b:b\zeta], \ for\ b \neq 0, b^3 \neq 1, \\ [1:\zeta:c], [1:c\zeta:c], \ for\ c \neq 0, c^3 \neq 1, \zeta \neq 1. \end{cases},$$

where ζ is a third root of 1. For computational purposes, we will only study the cases where $\zeta = 1$. Our approach to study $Z(D)$ is given in the following remark.

Remark V.16. We believe that we can employ the approach of the $|\sigma| = 3$ case. Namely, compute the center of D in degree 3 and 6, then show that the images of these elements under the surjection $D \twoheadrightarrow Skly_3$ are generators of $Z(Skly_3)$. However due to memory constraints, the computations with arbitrary parameters d_i, e_i cannot be executed at this time.

5.4 On the fat point modules of D

In the spirit of [Art92], [LB95b], and [SS93], we study fat point modules over D by presenting them as a quotient of a line module by a shifted line module. Although for generic D, we will see that we have to take other "curve modules" into consideration.

We restrict our attention to the cases where $|\sigma| = \infty$ (to say $D = D(1, b, c, d_i, e_i)$) and $|\sigma| = 2$ (to say $D = D(1, 1, c, d_i, e_i)$) as we have shown/ conjectured that such D are not PI (Proposition V.10, Conjecture V.13). (Here we assume by rescaling that the parameter a equals 1.) In fact, we have respective central subalgebras $k[\hat{g}_3, w]$ and $k[\hat{g}_2, w]$ for these central extensions, where $\deg(\hat{g}_d) = d$. We will use this data towards our aim to verify the nonexistence of fat point modules of D with $|\sigma| = 2, \infty$, or rather towards our aim to prove Conjecture V.9 for these cases.

Now given a fat point module F of D, our strategy to study F breaks down into four tasks:

I. Compute line modules M_l of D.

II. Compute central annihilators Ω_l for the line modules M_l.

In fact, we want our fat point module F to a share a central annihilator with some line module M_l. By [LB95b, proof of Proposition 10], the intersection of $\mathrm{ann}_D F$ and the graded central subalgebra $k[\hat{g}_d, w^d]$ is nonzero. Hence for every point $[\alpha : \beta] \in \mathbb{P}^1$, we require a line module $M_{\alpha\beta}$ with corresponding central annihilator $\Omega_{\alpha\beta} = \alpha w^d + \beta \hat{g}_d$.

Continuing with our strategy, we have the final two tasks:

III. Present F (say of multiplicity $\epsilon > 1$) by line modules. In other words, verify the existence of an exact sequence in D-qgr:

$$0 \to M_{l'}[-\epsilon] \to M_l \to F \to 0,$$

for line modules M_l and $M_{l'}$ of D.

IV. Use the geometry of the line scheme of D (Proposition V.4) to yield results about the structure/existence of the fat point module F.

Remark V.17. Karen Smith suggested that presenting fat point modules of D with plane modules (rather than line modules) may be useful. In particular, does there exists a sequence

$$0 \longrightarrow \bigoplus_{j=1}^{n-1} M_{H''_j} \longrightarrow \bigoplus_{i=1}^{n} M_{H'_i} \longrightarrow M_H \longrightarrow F \longrightarrow 0$$

for plane modules M_H, M_{H_i}, M_{H_j} of D? For now, we leave it to the reader to explore such an approach.

Case: $|\sigma| = \infty$

I. We begin by constructing a family of line modules for $D = D(1, b, c, d_i, e_i)$.

Lemma V.18. *Given a central extension $D = D(1, b, c, d_i, e_i)$, we have a 5 parameter family of line modules of D:*

$$\left\{ M_l \;=\; \frac{D}{D(x + u_2 y + u_3 z + u_4 w) + D(v_1 x + u_2 v_1 y + u_3 v_1 z + v_4 w)} \right\},$$

where $u_2, u_3, u_4, v_1, v_4 \in k$ and $(v_1, v_4) \neq (0, 0)$.

Proof. By [LBSVdB96, Proposition 4.1], there is a bijective correspondence between line modules of D:

$$\{D/(Du + Dv) \mid u, v \in D_1\},$$

and rank two tensors of D:

$$\{r \otimes u - sv \in R_D \mid r, s, u, v \in D_1\}$$

where R_D is the ideal of relations of D. With Affine, we have constructed a 5 parameter family of rank two tensors given by:

$$\begin{aligned} r &= (b_1v_1)\cdot x + (b_1u_2v_1)\cdot y + (b_1u_3v_1)\cdot z + (b_1v_4)\cdot w, \\ s &= b_1\cdot x + (b_1u_2)\cdot y + (b_1u_3)\cdot z + (b_1u_4)\cdot w, \\ u &= x + u_2\cdot y + u_3\cdot z + u_4\cdot w, \\ v &= v_1\cdot x + (u_2v_1)\cdot y + (u_3v_1)\cdot z + v_4\cdot w, \end{aligned}$$

with u_i, b_i, $v_i \in k$ for $i = 1,2,3,4$. The verification computation is Routine A.6 in the appendix. $\square$

II. Next, we need to compute the central annihilators Ω_l for the line modules $M_l = D/Du + Dv$ given in the lemma above, aiming to get a $\mathbb{P}^1$'s worth of pairs (M_l, Ω_l) with $\Omega_l \cdot M_l = 0$. Note that $\dim_k D_2 = 10$. Thus for $i = 1,2$, we require scalars $\{a_{i,j}\}$ so that

$$f_i = a_{i,1}x^2 + a_{i,2}xy + a_{i,3}xz + \cdots + a_{i,10}w^2 \in D_2 \quad \text{with} \quad f_1u + f_2v = \alpha w^3 + \beta\hat{g}_3$$

for some $[\alpha : \beta] \in \mathbb{P}^1$. This computation is unattainable at the moment due to memory restrictions, yet we proceed to step III assuming that this task has been achieved; see Assumption V.20 below.

III. Along with line modules, we also consider the following geometric modules.

Definition V.19. A **plane degree d curve module** of D is a cyclic graded left D-module M with Hilbert series:

$$H_M(t) = \frac{1 + t + \cdots + t^{d-1}}{(1-t)^2}.$$

Assumption V.20. Recall that for $|\sigma| = \infty$, we aim to prove that fat point modules over D do not exist via the use of the presentation of D-line modules. Moreover recall that if fat point modules of D, then we get that $\mathrm{ann}_D F \cap k[w^3, \hat{g}_3] \neq 0$. Hence there will be a central annihilator $\Omega_{\alpha\beta} = \alpha w^3 + \beta \hat{g}_3$ of F, and we need $\Omega_{\alpha\beta} \cdot M_l = 0$ for some D-line module M_l.

1. We assume that we have a pair $(M_{\alpha\beta}, \Omega_{\alpha\beta})$ of a line module with corresponding central annihilator for every point $[\alpha : \beta] \in \mathbb{P}^1$.
2. We assume that $\Omega_{\alpha\beta} \neq \Omega_{10} = w^3$. Otherwise F is a fat point module over $Skly_3$ which does not exist as $|\sigma| = \infty$ [ATVdB91, Theorem 7.5, Corollary 7.9]. Hence we are done in this case, i.e. we have verified Conjecture V.9.

As the central extension D is quadratic Auslander regular of global dimension 4 with Hilbert series $(1-t)^{-4}$ [LBSVdB96, Corollary 2.7], we employ the following result of LeBruyn relevant to D.

Proposition V.21. *[LB95b, Proposition 10] For a central extension D of $Skly_3$, assume that the following conditions hold.*

(H1) D has two central elements w^d and $\hat{g}_d$ of degree d.

(H2) For every $\Omega = \alpha w^d + \beta \hat{g}_d$ with $[\alpha : \beta] \in \mathbb{P}^1$, there is a line module $M_\Omega = D/DU$ with U a 2-dimensional subspace of D_1 such that:

(H2a) $\Omega \cdot M_\Omega = 0$;

(H2b) For every $u \in U$, $\Omega \cdot (D/Du) \neq 0$;

(H2c) M_Ω has a point module as a quotient.

Then every fat point module of D is either a quotient of a line module or of a plane degree $d-1$ curve module of D. □

We proceed by showing that the hypothesis of this proposition hold for D with $|\sigma| = \infty$.

Lemma V.22. *The hypotheses of Proposition V.21 hold for D with $|\sigma| = \infty$.*

Proof. Now (H1) holds by taking $d = 3$ (Equation (5.1)). Considering the family of line modules constructed in Lemma V.18, Assumption V.20 implies that (H2a) holds. By [LBSVdB96, Proposition 5.1.1(2)], we get that line modules $M = D/Du+Dv$ with $u, v \in D_1$ correspond to lines $l = \mathbb{V}(u,v) \subseteq \mathbb{P}^3$ in the line scheme of D. Recalling the structure of the line scheme (Proposition V.4), we know that M/Mw (corresponds to the point $\mathbb{V}(u,v,w) \in E$ and) is a point module of $Skly_3$, and thus of D. Hence condition (H2c) holds.

Finally we have that condition (H2b) holds as follows. Suppose not, i.e. that there exists $u \in U$ with $\Omega \cdot (D/Du) = 0$. Then $\Omega = du$ for some $d \in D_2$. In the domain, $B \cong D/(Dw + Dg)$, we have that $\overline{d}\overline{u} = \overline{\Omega} = 0$. So either $\overline{d}$ or $\overline{u}$ equals 0. In the first case $d = aw$ for some $a \in D_1$, and in the second case $u = bw$ for some $b \in k$. Both cases imply that w divides Ω, which contradicts Assumption V.20(2). $\square$

Therefore given a fat point module F of multiplicity $\epsilon > 1$, we have one of the presentations of F in D-qgr:

$$0 \to K \to M_l \to F \to 0 \qquad \text{or} \qquad 0 \to K \to N \to F \to 0,$$

where M_l is a line module of D and N is a plane curve degree 2 module of D. Since we do not have results on the kernel of $N \twoheadrightarrow F$ at this time, we make the following assumption.

Assumption V.23. Given a fat point module F of D, we assume that F is the quotient of a line module M_l of D.

Now we study the kernel of $M_l \twoheadrightarrow F$. Before we do so, consider the following terminology.

Definition-Lemma V.24. *[LBSVdB96, Corollary 2.7] Given a central extension D of $Skly_3$, let M be a finitely generated D-module.*

1. *The* **grade** *of M is the quantity $j(M) \in \mathbb{N} \cup \{+\infty\}$ defined by*

$$j(M) := \inf\{i \mid Ext^i_D(M, D) \neq 0\}.$$

2. *D is a* **Cohen-Macaulay (CM) ring** *in the sense that*

$$GKdim(M) + j(M) = 4 \ (= gldim(D)).$$

3. *The module M is called a* **Cohen Macaulay (CM) module** *if pd(M)=j(M), to say $Ext^i_D(M, D) = 0$ if $i \neq j(M)$.*

Proposition V.25. *Let F be a fat point module of D of multiplicity $\epsilon > 1$, which is the quotient of a line module M_l. Then we have that the kernel of $M_l \twoheadrightarrow F$ is isomorphic to a shifted line module $M_{l'}[-\epsilon]$.*

Proof. We assume that

$$0 \longrightarrow K \longrightarrow M_l \longrightarrow F \longrightarrow 0, \tag{5.3}$$

exact in D-qgr. Without loss of generality, we can assume that this sequence is exact in D-gr as F is equivalent to a submodule for which this condition holds. We now have to show that K is a shifted line module. Apply $\mathrm{Hom}_D(-, D)$ to sequence (5.3) and take cohomology. Let $E^i(P) :=\mathrm{Ext}^i_D(P, D)$ for a D-module P. We get the following long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_D(F,D) \arrow[r] & \mathrm{Hom}_D(M_l,D) \arrow[r] & \mathrm{Hom}_D(K,D) \arrow[dll] \\ & E^1(F) \arrow[r] & E^1(M_l) \arrow[r] & E^1(K) \ldots \end{tikzcd}$$

Since D is a Cohen Macaulay ring, we have that:

$$j(F) = GKdim(D) - GKdim(F) = 4 - 1 = 3.$$

Moreover line modules have GK-dimension 2, so we have that:

$$j(M_l) = GKdim(D) - GKdim(M_l) = 4 - 2 = 2.$$

Since $2 = j(M_l) = \inf\{j(K), j(F) = 3\}$ [LS93, remarks on p.43], we get that $j(K) = 2$. Now $E^i(M_l) = E^{i+1}(F) = 0$ for $i \neq 2$; hence $E^i(K) = 0$ for $i \neq 2 = j(K)$. Thus K is CM of GK-dimension 2 (which equals GKdim$D - j(K)$). Moreover we get that mult$(K) = 1$ as mult(M_l) =1.

It now remains to verify the following statement.

<u>Sublemma</u>. If K is a finitely generated graded D-module, CM, of GK-dimension 2 and multiplicity 1, then K is a shifted line module.

Proof of Sublemma. This proof is based on techniques of [LS93, §2]. We first shift the grading of K to assume that

$$K = K_0 \oplus K_1 \oplus K_2 \oplus \dots$$

with $K_0 \neq 0$. Moreover K is 2-critical due to [LS93, Lemma 1.11].

If w is a zero divisor of K, then we get that $wK = 0$ by [LS93, Lemma 2.10]. Thus K is a module over $Skly_3$ and is isomorphic to a shifted line module by [ATVdB91, Proposition 6.2].

If w is a nonzero divisor of K, then form $\overline{K} = K/wK$. Since $H_{\overline{K}}(t) = (1-t)H_K(t)$, we have that GKdim($\overline{K}$)=1 and mult($\overline{K}$)=1. Considering the sequence

$$0 \longrightarrow K \xrightarrow{\cdot w} K[1] \longrightarrow \overline{K}[1] \longrightarrow 0,$$

we have that the first two terms are CM, thus $\overline{K}$ is also CM [LS93, Lemma 1.12]. Hence by [LS93, Proposition 2.6],

$$H_{\overline{K}}(t) = t^p(1-t)^{-1} = t^p + t^{p+1} + t^{p+2} + \dots$$

for some $p \in \mathbb{Z}$. For $i \leq 0$:

$$\begin{aligned} \dim \overline{K}_i &= \dim K_i - \dim(wK)_i \\ &= \dim K_i - \dim K_{i-1} \qquad = \dim K_i, \end{aligned}$$

so $p = 0$. Thus $H_{\overline{K}}(t) = (1-t)^{-1}$ and so $H_K(t) = (1-t)^{-2}$. Now we must show that K is cyclic.

Let $K^\vee := E^2(K)$, the dual of K. Since $K \cong (K^\vee)^\vee$, it suffices to prove that $K^\vee$ is cyclic. By [LS93, Remark 1.12], if $P_\bullet \xrightarrow{\partial} K \to 0$ is a minimal resolution of K, then $P_\bullet^\vee \xrightarrow{\partial^\vee} K^\vee \to 0$ is a minimal resolution of $K^\vee$. More precisely, consider the following minimal resolution of K:

$$0 \longrightarrow \bigoplus_i D[-i]^{p_i} \xrightarrow{\partial_1} \bigoplus_i D[-i]^{m_i} \xrightarrow{\partial_0} \bigoplus_i D[-i]^{n_i} \longrightarrow K \longrightarrow 0$$

where n_i is the number of generators of K of degree i. Then, the sequence below is a minimal resolution of $K^\vee$:

$$0 \longrightarrow \bigoplus_i D[i]^{n_i} \xrightarrow{\partial_0^\vee} \bigoplus_i D[i]^{m_i} \xrightarrow{\partial_1^\vee} \bigoplus_i D[i]^{p_i} \longrightarrow K \longrightarrow 0$$

where p_i is the number of generators of $K^\vee$ of degree $-i$.

Since $K = \bigoplus_{i \geq 0} K_i$, we have that $n_i = 0$ for $i \leq -1$. Therefore $m_i = 0$ for $i \leq 0$ and $p_i = 0$ for $i \leq 1$. Now by [LS93, Proposition 1.10], the Hilbert series of $K^\vee$ is

$$H_{K^\vee}(t) \;=\; t^{-4}H_K(t^{-1}) \;=\; t^{-2} + 2t^{-1} + 3 + 4t + 5t^2 + \dots.$$

Thus $p_2 = 1$ and $p_i = 0$ for $i \geq 3$. Hence $K^\vee$ is cyclic with a generator in degree -2, and so K is cyclic as desired. This concludes the proof of the sublemma.

Thus K is a shifted line module over D with $H_K(t) = t^d(1-t)^{-2}$. This implies that

$$H_F(t) = H_{M_l}(t) - H_K(t) = (1 + t + \cdots + t^{d-1})(1-t)^{-1}.$$

Hence $\epsilon = \mathrm{mult}(F) = d$ and $K \cong M_{l'}[-\epsilon]$ for some line module $M_{l'}$ over D. □

IV. Although we do not have a conclusion about the existence/ structure of fat point modules for D with $|\sigma| = \infty$, we do have the following geometric result involving the line modules of D, which is an adaptation of [ATVdB91, Propositions 6.23, 6.24].

Proposition V.26. *Let $D = D(1, b, c, d_i, e_i)$ be a central extension of $Skly_3$. For a fat point module F of multiplicity $\epsilon > 1$, consider the exact sequence*

$$0 \to M_{l'}[-\epsilon] \to M_l \to F \to 0$$

resulting from Proposition V.25, where M_l and $M_{l'}$ are line modules over D. Then we have that $l' \cap E = \sigma^{-\epsilon}(l \cap E)$.

Proof. We use techniques from [ATVdB91, §6]. Say $M_l = D/(Du + Dv)$ for some $u, v \in D_1$ and so $l = \mathbb{V}(u, v) \subseteq \mathbb{P}^3$ [LBSVdB96, Proposition 5.1.1]. Let P be the scheme theoretic intersection of l and E which is nonempty due to [LBSVdB96, page 204]. Let $M(P) = (\Gamma_*(\mathcal{O}_P))_{\geq 0}$ where $\Gamma_n(\mathcal{O}_P) = H^0(E, \mathcal{O}_P \otimes \mathcal{L}_n)$. This is a B-module. Taking Γ_* of $\mathcal{O}_E \to \mathcal{O}_P$ yields $B \to M(P)$, surjective in high degree. Now we get the map ϕ as in the following diagram:

$$\begin{tikzcd} & & B = D/(w,g) \arrow[dr] & & \\ 0 \arrow[r] & K(P) \arrow[r] & D \arrow[ur] \arrow[rr, dashed, "\phi"] & & M(P) \arrow[r] & 0 \end{tikzcd}$$

with the horizontal sequence exact in D-qgr. Thus $M(P)$ is equivalent to $D/K(P)$ in D-qgr. Since P is a 0-dimensional subscheme of E, let P=Spec R for some finite

k-algebra R. Hence

$$R \;\cong\; H^0(E, \mathcal{O}_{\mathrm{Spec}R}) \;=\; H^0(E, \mathcal{O}_P \otimes \mathcal{L}_0) \;=\; M(P)_0.$$

This implies that $\phi(1_D) = 1_R$. Now

$$P \subseteq l = \mathbb{V}(u,v) \iff u \cdot 1_R = v \cdot 1_R = 0 \text{ in } M(P) \iff u, v \in K(P).$$

Hence $\phi : D \to M(P)$ factors through the D-gr surjection $D \twoheadrightarrow M_l$ to get the map ψ below:

$$\begin{tikzcd} 0 \arrow[r] & K(P) \arrow[r] & D \arrow[rr, "\phi"] \arrow[dr] & & M(P) \arrow[r] & 0 \\ & & & M_l \arrow[ur, dashed, "\psi"'] & & \end{tikzcd}$$

As ϕ is surjective in D-qgr, the cokernel of ϕ is finite dimensional. Thus cokerψ is also finite dimensional. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & M_{l'}[-\epsilon] \arrow[r] & M_l \arrow[r] \arrow[d, "\psi"] & F \arrow[r] & 0 \\ & & M(P) & & \end{tikzcd}$$

Now $\ker\psi \nsubseteq M_{l'}[-\epsilon]$; else we get the induced map $F \to M(P)$ which implies that $F/(\ker(F \to M(P)))$, equivalent to a submodule of $M(P)$. So the fat point module F would be a B-module, a contradiction. Note that $M_l/(\ker\psi + M_{l'}[-\epsilon])$ is finite dimensional due to F being 1-critical. Consider the map $\psi|_{M_{l'}[-\epsilon]}$ and we get that:

$$\begin{aligned} \mathrm{coker}(\psi|_{M_{l'}[-\epsilon]}) &\cong \frac{M(P)}{\mathrm{im}(\psi|_{M_{l'}[-\epsilon]})} \\ &\cong \frac{M(P)}{M_{l'}[-\epsilon]/(\ker\psi \cap M_{l'}[-\epsilon])} \\ &\cong \left(\frac{M_l}{\ker\psi}\right) \Big/ \left(\frac{\ker\psi + M_{l'}[-\epsilon]}{\ker\psi}\right), \end{aligned}$$

and the latter is finite dimensional. Therefore $\psi|_{M_{l'}[-\epsilon]} : M_{l'}[-\epsilon] \to M(P)$ has finite cokernel. By shifting we get that $M_{l'} \to M(P)[\epsilon] \cong M(\sigma^{-\epsilon}P)$ and this map has a finite cokernel. Thus l' contains $\sigma^{-\epsilon}P$ and $\sigma^{-\epsilon}P = (\sigma^{-\epsilon}P) \cap E \subseteq l' \cap E$, which implies that scheme-theoretically: $l' \cap E = \sigma^{-\epsilon}P$. □

Case: $|\sigma| = 2$

Now we analyze fat point modules of $D(1,1,c,d_i,e_i)$ via the steps I - IV stated at the beginning of the section. Most results will follow in a similar fashion to the $|\sigma| = \infty$ case. However a conclusion about the existence/ structure of fat point modules for these central extensions D remains unknown.

I. We begin by constructing a family of line modules.

Lemma V.27. *Given a central extension $D(1,1,c,d_i,e_i)$, we have a 5 parameter family of line modules of D:*

$$\left\{ M_l \ = \ \frac{D}{D(x+u_2y+u_3z+u_4w)+D(v_1x+u_2v_1y+u_3v_1z+v_4w)} \right\},$$

where $u_2, u_3, u_4, v_1, v_4 \in k$.

Proof. Referring to the proof of Lemma V.18, note that the line modules do not depend on the parameter b. Hence we obtain the same family of line modules $\{M_{u_2u_3u_4v_1v_4}\}$ for $D(1,1,c,d_1,d_2,d_3,e_1,e_2,e_3)$. □

II. For D with $|\sigma| = 2$, Assumption V.20 is not needed as we have the following result.

Lemma V.28. *For $D = D(1,1,c,d_i,e_i)$, the line modules $\{M_{u_2u_3u_4v_1v_4}\}$ have central annihilators $\Omega_l = \hat{g}_2 + \gamma w^2$ where γ depends on scalars u_4, v_1, v_4. Thus we get a pair $(M_{\alpha\beta},\ \Omega_{\alpha\beta})$ of a line module and corresponding central annihilator for each $[\alpha:\beta] \in \mathbb{P}^1$.*

Proof. If $[\alpha : \beta] = [1 : 0]$, then $\Omega_{\alpha\beta} = w^3$. Moreover we have line modules M over $Skly_3$, which exist by Example II.20, so that $w^3 \cdot M = 0$. On the other hand, if $[\alpha : \beta] \neq [1 : 0]$, then $[\alpha : \beta] = [\gamma : 1]$ for some $\gamma \in k$ and $\Omega_{\alpha\beta} =: \Omega_\gamma = \gamma w^2 + \hat{g}_2$. See Routine A.7 in the appendix for the proof that for all $\gamma \in k$, we have that Ω_γ annihilates some line module $M_{u_2 u_3 u_4 v_1 v_4}$. □

III. Now we have that fat point modules of $D = D(1,1,c,d_i,e_i)$ are presented by line modules of D by the following argument.

Proposition V.29. *Let $|\sigma| = 2$, so we consider the central extension $D = D(1,1,c,d_i,e_i)$ over $S(1,1,c)$. Take F a fat point module of D of multiplicity $\epsilon > 1$. Then there exist line modules M_l and $M_{l'}$ for which $0 \to M_{l'}[-\epsilon] \to M_l \to F \to 0$ is exact in D-qgr.*

Proof. First note that we do not require Assumption V.23 as Proposition V.21 holds for $d = 2$. In other words, we always get that F is the quotient of a line module (as a plane curve degree 1 module is a line module by definition). The rest of the proof follows from the proof for Proposition V.25. □

IV. Moreover we have the same geometric result (Proposition V.26) from the $|\sigma| = \infty$ case, and its proof follows in the same manner.

Proposition V.30. *Let $D = D(1,1,c,d_i,e_i)$ be a central extension of $S(1,1,c)$. For a fat point module F of D of multiplicity $\epsilon > 1$, consider the exact sequence $0 \to M_{l'}[-\epsilon] \to M_l \to F \to 0$ resulting from Proposition V.29, where M_l and $M_{l'}$ are line modules over D. Then we have that $l' \cap E = \sigma^{-\epsilon}(l \cap E)$.* □

Thus for $|\sigma| = 2$, we understand D-fat point modules in terms of D-line modules and have some results about the geometry of such modules.

APPENDIX

APPENDIX A

Computational results, Affine and Maple routines

This appendix contains computational results and computer routines pertaining to the order of σ for the three-dimensional Sklyanin algebras $Skly_3 = S(a,b,c)$ (Definition I.2), to the 1-dimensional representations of deformed Sklyanin algebras S_{def} (Definition I.12), and to the line modules over central extensions D of $Skly_3$ (Definition IV.6).

Note:
We assume that the base field k is \mathbb{C} for these computations.

The automorphisms σ of finite order

Recall from §2.1.3 that the three-dimensional Sklyanin algebras $S = S(a,b,c)$ come equipped with geometric data $(E, \mathcal{L}, \sigma)$ where $E = E_{abc} \subseteq \mathbb{P}^2$ is given by:

$$E = E_{abc} : \mathbb{V}\left((a^3+b^3+c^3)xyz - (abc)(x^3+y^3+z^3)\right) \overset{i}{\subset} \mathbb{P}^2,$$

and $\mathcal{L}$ is the invertible sheaf $i^*\mathcal{O}_{\mathbb{P}^2}(1)$ on E. Here we assume that E is smooth, i.e. E is either $\mathbb{P}^2$ or an elliptic cubic curve. Moreover $\sigma = \sigma_{abc}$ is an automorphism of E induced by the shift functor on point modules of S whose image is the cross product of any two rows of the matrix $\mathtt{M_i}$ from Equation (2.1) [ATVdB90, §1]. In other words, σ can be explicitly given as follows:

$$\sigma([x:y:z]) = [acy^2 - b^2xz \ : \ bcx^2 - a^2yz \ : \ abz^2 - c^2xy]. \tag{A.1}$$

Now the behavior of $S(a,b,c)$ varies according to $|\sigma_{abc}|$. More precisely, $S(a,b,c)$ is module-finite over its center if and only if $|\sigma_{abc}| < \infty$ [ATVdB91, Theorem 7.1]. Hence the following proposition yields the first steps in the classification of the parameters (a,b,c) of $Skly_3$ for which the automorphism σ_{abc} of smooth E_{abc} have finite order. Since $S(a,b,c) \cong S\left(1, \frac{b}{a}, \frac{c}{a}\right)$, we assume that $a = 1$.

Proposition A.1. *Let ζ be a third root of unity. Then we have the following results.*

- $|\sigma| = 1 \iff [a:b:c] = [1:-1:0]$, *the origin of* E.
- $|\sigma| = 2 \iff [a:b:c] = [1:1:c]$, *for* $c \neq 0$, $c^3 \neq 1$.

Correction (pointed out by Daniel Reich):
For |\sigma|=n \leq 3,
only the implications <= hold.
We actually have more
parameters [a:b:c] so that
\sigma_{a,b,c} has order n.

- $|\sigma| = 3 \iff [a:b:c] = \begin{cases} [1:0:\omega], \ \text{for } \omega = -1, e^{\pi i/3}, e^{5\pi i/3}; \\ [1:\omega:0], \ \text{for } \omega = e^{\pi i/3}, e^{5\pi i/3}. \end{cases}$.

- $|\sigma| = 4 \iff [a:b:c] = \begin{cases} \left[1 : b : \left(\frac{b(b^2+1)}{b+1}\right)^{1/3}\zeta\right], \ \text{for } b \neq 0, -1, \pm i, \text{ and } b^3 \neq 1; \\ \left[1 : \left(\frac{f(c)^{2/3} - 12(1-c^3)}{6f(c)^{1/3}}\right)\zeta : c\right], \ \text{for } c \neq 0, c^3 \neq 1. \end{cases}$

  where $f(c) = 108c^3 + 12(12 - 36c^3 + 117c^6 - 12c^9)^{1/3}$.

- $|\sigma| = 5 \iff [a:b:c] = \begin{cases} \left[1 : b : \left(\frac{r \pm s^{1/2}}{2b}\right)^{1/3}\zeta\right], \ \text{for } b \neq 0, \ b^3 \neq 1; \\ \qquad\qquad b \neq \text{ primitive } 10^{th} \text{ root of unity}, \\ [1 : g(c) : c], \ \text{for } c \neq 0, c^3 \neq 1. \end{cases}$

  where r = $-b^5 + b^4 + b^3 + b^2 + b - 1$ *and* $s = (b^2 - 3b + 1)(b-1)^2(b^2 + b - 1)^3$.

  Moreover $g(c)$ *is a root of*

  $$Z^6 + (c^3 - 1)Z^5 + (1 - c^3)Z^4 + (-1 - c^3)Z^3 + (1 - c^3)Z^2 + (c^6 - c^3)Z + c^3.$$

- $|\sigma| = 6 \iff [a:b:c] = \begin{cases} [1:b:\zeta], [1:b:b\zeta], \ \text{for } b \neq 0, b^3 \neq 1, \\ [1:\zeta:c], [1:c\zeta:c], \ \text{for } c \neq 0, c^3 \neq 1, \zeta \neq 1. \end{cases}$

Proof. Refer to Routine A.3. □

Remark A.2. Although there is no apparent pattern for the parameters $(1, b, c)$ of a given $|\sigma| = n$, we believe that there are the following restrictions of b and c for $n \geq 4$.

1. When b is arbitrary and we solve for c, we must have that $b \neq 0$ and $b^3 \neq 1$. Moreover for
   - $4|n$: $b \neq$ primitive n^{th} root of 1,
   - n even, $4 \not| n$: $b \neq$ primitive $\left(\frac{n}{2}\right)^{th}$ root of 1,
   - n odd: $b \neq$ primitive $(2n)^{th}$ root of 1.
2. When c is arbitrary and we solve for b, we must have that $c \neq 0$ and $c^3 \neq 1$.

Computer Routines

Affine / Maple routines are given as follows:

```
### AFFINE / MAPLE ###

Sample routine

## Sample comment
```

Routine A.3. This is the proof of Proposition A.1.

```
### MAPLE ###

## Let |sigma| =:n, and take n>1. Make sure to change n as needed!

n:= 6;

## Let us define sigma.

p:=(x,y,z)->a*c*y^2-b^2*x*z:

q:=(x,y,z)->b*c*x^2-a^2*y*z:
```

```
r:=(x,y,z)->a*b*z^2-c^2*x*y:

x[0]:=x:     y[0]:=y:     z[0]:=z:

## We assume that a=1.

a:=1:


## For a given n, let x[n], y[n], z[n] be the entries of the image

##   of sigma^n. Let us define sigma^n.

for k from 0 to n-1 do

x[k+1]:=p(x[k],y[k],z[k]);

y[k+1]:=q(x[k],y[k],z[k]);

z[k+1]:=r(x[k],y[k],z[k]);

end do:


## We want (x[n]:y[n]:z[n]) = (x:y:z) projectively.

## So we need f=y*x[n]-x*y[n], g=z*x[n]-x*z[n], h=z*y[n]-y*z[n]

##   to be zero for some parameters (a,b,c), subject to the

##   defining relation of E.

f:=y*x[n]-x*y[n]:   g:=z*x[n]-x*z[n]:    h:=z*y[n]-y*z[n]:

E:=((1+b^3+c^3)/(b*c))*(x*y*z)-x^3-y^3:

## We will replace the term z^3 with the expression E.

d[1]:=2^n+1:


## Special care is needed for the n=2 and n=3 cases.

## In general, reduce f,g,h with respect to the relation of E.

## For n>6 we reduce f first, then g and h later.
```

```
for k from 1 to n-1 do
f:= expand(f-(add( coeff(f,z^i)*z^i, i=3..d[k]+1 ))
    + add( coeff(f,z^j)*z^(j mod 3)*E^(iquo(j,3)), j=3..d[k]+1));
g:= expand(g-(add( coeff(g,z^i)*z^i, i=3..d[k]+1 ))
    + add( coeff(g,z^j)*z^(j mod 3)*E^(iquo(j,3)), j=3..d[k]+1));
h:= expand(h-(add( coeff(h,z^i)*z^i, i=3..d[k]+1 ))
    + add( coeff(h,z^j)*z^(j mod 3)*E^(iquo(j,3)), j=3..d[k]+1));
d[k+1]:= iquo(d[k],3) + irem(d[k],3);
end do:


## Now the degree of f,g,h with respect of z should be <3.
degree(f,z);      degree(g,z);      degree(h,z);


## We can either: (I) Let b be arbitrary and solve for c, or
##   (II) Let c be arbitrary and solve for b
## Here is the computation for (I):
for k from 1 to 1 do
if type(n,odd) then
v:=solve(coeff(f,x^(d[1]))=0,c);
w:=solve(coeff(coeff(f,x^(d[1]-3)),y^3)=0,c);
else
v:=solve(coeff(coeff(f,x^(d[1]-1)),y)=0,c);
w:=solve(coeff(coeff(f,x^(d[1]-4)),y^4)=0,c);
end if     end do:
```

```
## Here is the computation for (II):

for k from 1 to 1 do

if type(n,odd) then

v:=solve(coeff(f,x^(d[1]))=0,b);

w:=solve(coeff(coeff(f,x^(d[1]-3)),y^3)=0,b);

else

v:=solve(coeff(coeff(f,x^(d[1]-1)),y)=0,b);

w:=solve(coeff(coeff(f,x^(d[1]-4)),y^4)=0,b);

end if     end do:


## We create a list of potential nonzero unique solutions

lenv:=nops([v]):     lenw:=nops([w]):

u:=[]:

for i from 1 to lenv do

for j from 1 to lenw do

t:=evalb(v[i]=w[j]):

if t=true then

if v[i] <> 0 then

u:=[op(u),v[i]];

end if     end if     end do:     end do:

lenu:=nops(u):

with(ListTools):     L:=MakeUnique(u);     lenL:=nops(L);


## So L is the list of solutions for which we need verify

##   that f,g,h (now reduced by the relation of E) are 0
```

```
## Case (I), letting b be arbitrary we use the subroutine:
for k from 1 to lenL do
c:=expand(L[k]):     print(simplify(c));
newf:=expand(f):     newg:=expand(g):     newh:=expand(h):
print(k, simplify(newf), simplify(newg), simplify(newh));
end do:
## (II) For c arbitrary we use the subroutine:
for k from 1 to lenL do
b:=expand(L[k]):     print(simplify(b));
newf:=expand(f):     newg:=expand(g):     newh:=expand(h):
print(k, simplify(newf), simplify(newg), simplify(newh));
end do:


## If the output is "0,0,0", then L is the desired list of
##   parameters (1,b,c) so that sigma(1bc)^n = 1.
## Eliminate by hand parameters (1,b,c) for which
##   sigma(1bc)^m = 1 with m<n.
-------------------
## ORDER 2 RESULTS:
## Compute f,g,h (modulo the relation of E) for sigma^2 as described
##   at the beginning of the routine. In particular, we want
##   (reduced) f to equal zero for some parameters (1,b,c).
> collect(f,[z,y,x]);
       ((c - b^3  c) y^3  + (b^2  c - b^5  c) x^3 ) z^2 + [...]
```

```
## Consider the coefficient of y^3*z^2: c(b^3-1).

## If equal to zero, then c=0 or b^3=1.

## If c=0 and b is arbitrary, then either E is not smooth

##   or |sigma|=3. Therefore we take c nonzero.


## First let b=1. Then we get that c arbitrary yields a solution.

> L:=MakeUnique(u);

                                L := [c[1]]

> b:=1;

                                   b := 1

> (simplify(f),simplify(g),simplify(h));

                                  0, 0, 0

## Now let b:=exp(2*Pi*I/3);

> L:=MakeUnique(u);

       L := [1, - 1/2 + 1/2 I 3^(1/2), - 1/2 - 1/2 I 3^(1/2)]

## so c^3=1, and S is not a Sklyanin algebra.

## Likewise result for b:=exp(4*Pi*I/3);

## Therefore the only solution is (1,b,c) = (1,1,c)

##   for c nonzero, not a third root of 1.

-------------------

## ORDER 3 RESULTS:

## (I) b nonzero and arbitrary, solving for c:

> L:=MakeUnique(u);     lenL:=nops(L);

                    L := []     lenL := 0

## (II) same as (I)
```

```
## If b=0, then E is a triangle, unless c = 3rd root of -1.

## Vice versa for c=0.

## For all six cases:

> (simplify(f),simplify(g),simplify(h));

                                   0, 0, 0

## Now we can conclude the result in the case of |sigma|=3.

-------------------

## ORDER 4,5,6 RESULTS for subroutines (I, II) are given in Prop. A.1
```

Routine A.4. We explicitly list the 1-dimensional representations of $S_{def}(a,b,c,d_i,e_i)$ for each of the cases: $d_i = 0$, $e_i = 0$, and $d_2 = d_3 = e_2 = e_3 = 0$. We use the following short list of commands in Maple, and by rescaling we assume that the parameter a is 1.

```
### MAPLE ###

a:=1;

f:=a*y*z + b*z*y + c*x^2 + d1*x + e1;

g:=a*z*x + b*x*z + c*y^2 + d2*y + e2;

h:=a*x*y + b*y*x + c*z^2 + d3*z + e3;

v:=solve({f=0, g=0, h=0}, [x,y,z]);
```

For $d_i = 0$, there is one 1-dimensional representation:

```
v := [[x = (-e2 c e1^2  b^2  - 2 e2 c e1^2  b {...}

            - 4 %1^2  e2^2  b)  /  ((1 + b) (%1^2  c^3  e2 {...}

            + c e1^2  b^2  + c e1^2 ) %1),

       y = - %1 (2 e3 c {...}  + 2 %1^2  c^3  e3 e2) /
```

```
          (-e2 c e1^2  b^2  {...}  - 4 %1^2  e2^2  b),  z = %1]]


%1 := RootOf((c + 3 c^4  + 3 c^7  +  {...} + c^2  e2^2  e1^2 )
```

For $e_i = 0$, there are five 1-dimensional representations:

```
v := [[x = 0, y = - d2/c, z = 0],

       [x = 0, y = 0, z = - d3/c],

       [x = 0, y = 0, z = 0],

       [x = - d1/c, y = 0, z = 0],


       [x = - (%1 d1 b^3 {...} d3) / (c (%1 d3 b^3  {...} - d1^2)^2),

        y = %1,

        z = (-%1^2  b^4  d1 - {...} d2 d3^2 c^2)/ (c {...} - d1^2)]]


%1 := RootOf((3 c^3  +  {...} + 5 d3^2 b^4 d1^2)
```

For $d_2 = d_3 = e_2 = e_3 = 0$, there are three 1-dimensional representations:

```
v := [[x = RootOf(c _Z^2  + d1 _Z + e1), y = 0, z = 0],

       [x = %2 c, y = -%2 (1 + b), z = -%2 (1 + b)],


    [x = ((%1 b^3  + {...} + c d1) c %1^2) / (e1 + %1 c d1),

     y = - (e1 b + %1 b c d1 + e1 + %1 c d1)/(%1 b^3 + {...} + c d1),

     z = %1 (1 + b)]]


%1 := RootOf((2 c^3 + {...} + c d1) _Z^3 , label = _L2)

%2 := RootOf((1 + {...} + c^3) _Z^2 + e1 + c d1 _Z, label = _L1)
```

Routine A.5. Here we compute the point scheme of a central extension D of a three-dimensional Sklyanin algebra. A detailed point scheme computation in provided in Proposition II.15 for example. Recall the definition of a d^{th} truncated point scheme V_d (Definition II.4) and the results of Definition-Lemma II.6. Furthermore, note that V_1 for D is isomorphic to $\mathbb{P}^3$. Considering the multilinearizations of relations of D, we get that $\mathtt{M_i} \cdot [x_{i+1} : y_{i+1} : z_{i+1} : w_{i+1}]^T = 0$ for

$$\mathtt{M_i} = \begin{pmatrix} cx_i & bz_i & ay_i & d_1x_i + e_1w_i \\ az_i & cy_i & bx_i & d_2y_i + e_2w_i \\ by_i & ax_i & cz_i & d_3z_i + e_3w_i \\ -w_i & 0 & 0 & x_i \\ 0 & -w_i & 0 & y_i \\ 0 & 0 & -w_i & z_i \end{pmatrix}.$$

To show that the point scheme data (V_d, π_d) of D stabilizes at $d = 2$, we must verify that rank$(\mathtt{M_i})\geq 3$ for all $i \geq 1$. Suppose not, then we get from the routine below that $x_i = y_i = z_i = w_i = 0$, a contradiction. Here we drop the index i.

```
### MAPLE ###

with(LinearAlgebra):

M:= << c*x | b*z   | a*y    | d1*x+e1*w >,

     < a*z | c*y   | b*x    | d2*y+e2*w >,

     < b*y | a*x   | c*z    | d3*z+e3*w >,

     < -w  | 0     | 0      | x         >,

     < 0   | -w    | 0      | y         >,

     < 0   | 0     | -w     | z         >>;


## Create a list of the 4X4 full submatrices of M (in row notation).
```

```
## There are 6 choose 4 (so 15) of them.

R:=[];

for i from 1 to 6 do

R:=[op(R),Row(M,i)]:

end do:

with(combinat):

S:=choose(R,4):


## Convert the matrices in row notation into the "Matrix" format.

L:=[];     L1:=[];

for n from 1 to 15 do

for i from 1 to 4 do

A[i]:=convert(S[n][i],Matrix):

end do:

B:=<Transpose(A[1])|Transpose(A[2])|Transpose(A[3])|Transpose(A[4])>:

C:=Transpose(B):


## Then for each of the 15 matrices,

##   we compute the 3x3 minors and make a list of them.

for i from 1 to 4 do

for j from 1 to 4 do

m:=Minor(C,i,j):

L:=[op(L),m]:

end do:     end do:     end do:
```

```
## We set the 240 3x3 minors equal to 0, then solve the system.
for i from 1 to 240 do
L1:=[op(L1),L[i]=0]:
end do:
v:=solve(L1,[x,y,z,w]);
                      v := [[x = 0, y = 0, z = 0, w = 0]]
```

Hence if all of the $3{\times}3$ minors of $\mathtt{M_i}$ vanish, then we get a contradiction. Thus rank($\mathtt{M_i}$)≥ 3 for $i \geq 1$. Now given the map $\pi_2 : V_2 \to \mathbb{P}^3$, we know that the point scheme P_D of D is isomorphic to the graph of the image of π_2. Call this image X and we have that

$$X = \{[x_1 : y_1 : z_1 : w_1] \in \mathbb{P}^3 \mid \operatorname{rank}(\mathtt{M_1}) < 4\}.$$

To compute X, we need solutions $[x_1 : y_1 : z_1 : w_1] \in \mathbb{P}^3$ so that determinants of the full 4×4 submatrices of $\mathtt{M_1}$ vanish. The following is the Maple routine to achieve such a result; the solutions are explicitly provided in the cases where the parameters d_i or e_i are 0. Here we drop the index 1.

```
### MAPLE ###
## (Optional) d1:= 0; d2:= 0; d3:= 0;  or e1:= 0; e2:= 0; e3:= 0;
##          or d2:= 0; d3:= 0; e2:= 0; e3:= 0;
## Define the matrix M1.
with(LinearAlgebra):
M:= << c*x | b*z   | a*y    | d1*x+e1*w >,
     < a*z | c*y   | b*x    | d2*y+e2*w >,
     < b*y | a*x   | c*z    | d3*z+e3*w >,
     < -w  | 0     | 0      | x         >,
```

```
     < 0   | -w    | 0      | y         >,
     < 0   | 0     | -w     | z         >>;


## Create a list of the 4X4 full submatrices of M (in row notation).
## There are 6 choose 4 (so 15) of them.
R:=[];
for i from 1 to 6 do
R:=[op(R),Row(M,i)]:
end do:
with(combinat):
S:=choose(R,4):


## Convert the matrices in row notation into the "Matrix" format.
## Now for each matrix in the list, compute its determinant.
## Create a list of the 15 determinants excluding
##   redundancies and zeros. Call this list T.
t:=[]:
for n from 1 to 15 do
for i from 1 to 4 do
A[i]:=convert(S[n][i],Matrix):
end do:
B:=<Transpose(A[1])|Transpose(A[2])|Transpose(A[3])|Transpose(A[4])>:
C:=Transpose(B):
d:=Determinant(C);
if d<>0 then
```

```
t:=[op(t),d]:
end if:     end do:
with(ListTools):
T:=MakeUnique(t):


## Set each entry of list T equal to 0, and solve the system
##   of equations for variables {x,y,z,w}.
W:=[];
for j from 1 to nops(T) do
W:=[op(W),T[j]=0]:
end do:
v:=solve(W,[x,y,z,w]);


## If the solution includes "w=0", then this data corresponds
##   to the point scheme of Skly3.
---------------------------------------------------------
## for d1 = d2 = d3 = 0:
[x = RootOf(c _Z^3 a b + (-y a^3 z - y c^3 z - y b^3 z)_Z
     + a z^3 b c + y^3 b c a),
 y = y, z = z, w = 0]


[x = 0, y = -z, z = z, w = 0]
[x = 0, y = RootOf(_Z^2  + 1 - _Z, label = _L1) z, z = z, w = 0]


[x = - z (c^3 e2^3 - {...} + e2 b^3 e3^2 %1) /
```

```
     (c^2 (e2^3  - e3^3 ) %1 (a + b)),

 y = %1 z, z = z,

 w = RootOf((e2^3 - e3^2 ) _Z^2 - {...} + %1^3 c^3 e3^2 ,

     label = _L4) z/c]


%1 := RootOf((c e1 a^2 e2 +  {...} - a^3 e2^2) _Z^3 ,

      label = _L2)
```

Thus in the case $d_i = 0$, the cardinality of the set $\mathcal{S}$ is 1. In fact,

$$\mathcal{S} = \left\{ \left[\frac{c^3 + \{\ldots\}}{c^2(e_2^3 - \{\ldots\})} \; : \; R \; : \; 1 \; : \; \sqrt{e_2^3 + \{\ldots\}} \right] \right\}$$

where R is the "Rootof" expression above.

```
------------------------------------------------------------

## for e1 = e2 = e3 = 0:

[x = RootOf(c _Z^3 a b + (-y b^3 z - y a^3 z - y c^3 z) _Z

     + y^3 b c a + a b z^3 c),

y = y,  z = z,  w = 0]


[x = 0, y = -z, z = z, w = 0]

[x = 0, y = RootOf(_Z^2  + 1 - _Z, label = _L1) z, z = z, w = 0]

[x = 0, y = - d2 w / c, z = 0, w = w]

[x = 0, y = 0, z = 0, w = w]

[x = - d1 w / c, y = 0, z = 0, w = w]

[x = 0, y = 0, z = z, w = - c z / d3]


[x = %1 z c (-d2 + %1 d3) / (d2 %1^2  - d3) (a + b),
```

```
y = %1 z,  z = z,  w = - z (%1^3  - 1) c / d2 %1^2  - d3]


%1 := RootOf((a^3 d2^2 + {...} - c^2 a d1 d3) _Z^4 {...}

     + (c^2 b d1 d2 + c^2 a d1 d2 + c^3 d3^2 ) _Z^3 , label = _L2)
```

Thus in the case where $e_i = 0$, the cardinality of the set $\mathcal{S}$ is 5. In fact,

$$\mathcal{S} = \left\{[0:0:0:1],\ \left[\frac{-d_1}{c}:0:0:1\right],\ \left[0:\frac{-d_2}{c}:0:1\right],\ \left[0:0:\frac{-d_3}{c}:1\right],\right.$$

$$\left.\left[\frac{Rc(-d_2+Rd_3)}{(d_2R^2-d_3)(a+b)}\ :\ R\ :\ 1\ :\ \frac{(R^3-1)c}{d_2R^2-d_3}\right]\right\}$$

where R is the "Rootof" expression above.

```
------------------------------------------------------------

## for d2 = d3 = e2 = e3 = 0:

v := [[x = RootOf(c _Z^3 b + (-b^3 y z - y z - c^3 y z) _Z

           + b y^3 c + b z^3 c),

       y = y, z = z, w = 0],


      [x = 0, y = -z, z = z, w = 0],

      [x= 0, y = RootOf(_Z^2 + 1 - _Z, label = _L1) z, z= z, w= 0],


      [x= x, y=0, z=0, w= RootOf(e1 _Z^2 + c + _Z d1, label= _L2)x],


      [x = x, y = - x (b + 1) /c, z = - x (b + 1) /c,

       w = RootOf(e1 _Z^2 + {...} + d1c _Z, label = _L4) x /c],


      [x =((b^3  + {...} + 1) c z) /((%1 b e1 + {...} + c d1) %1),
```

```
        y = - (e1 %1 + c d1) z %1 / (b^3 + {...} + 1),

        z = z, w = %1 z / (b+1)]]


%1 := RootOf(e1^2 _Z^4 + {...} + _Z^3 c d1 e1, label = _L6)
```

Thus in the case where $d_2 = d_3 = e_2 = e_3 = 0$, the cardinality of the set $\mathcal{S}$ is 3. We can interpret the points of $\mathcal{S}$ as in the previous cases.

Routine A.6. We verify the construction of the rank two tensors of $D(1, b, c, d_1, d_2, d_3, e_1, e_2, e_3)$ (and hence of $D(1, 1, c, d_1, d_2, d_3, e_1, e_2, e_3)$ used in the proof of Lemma V.18 (and Lemma V.27) in §5.4.

```
### AFFINE ###

## Define the algebra D(1,b,c,d_1,d_2,d_3,e_1,e_2,e_3)

declare(c,constant,d1,constant,d2,constant,d3,constant,

e1,constant,e2,constant,e3,constant);

declare_weights(x,1,y,1,z,1,w,1);

ALL_DOTSIMP_DENOMS:[];

r1:y.z+b*z.y+c*x.x+d1*x.w+e1*w.w;

r2:z.x+b*x.z+c*y.y+d2*y.w+e2*w.w;

r3:x.y+b*y.x+c*z.z+d3*z.w+e3*w.w;

r4:x.w-w.x;    r5:y.w-w.y;    r6:z.w-w.z;

set_up_dot_simplifications([r1,r2,r3,r4,r5,r6],7);

y n n n     [x,y,z,w];


## Define the rank two tensor expressions au-bv

declare(b1,constant,u2,constant,u3,constant,u4,constant,
```

```
v1,constant,v4,constant);

a1: b1*v1;    a2: v1*b1*u2;    a3: b1*v1*u3;   a4: v4*b1;

b2: b1*u2;    b3: b1*u3;       b4: b1*u4;

u1: 1;        v2: u2*v1;       v3: v1*u3;

r: a1*x+a2*y+a3*z+a4*w;      s: b1*x+b2*y+b3*z+b4*w;

u: u1*x+u2*y+u3*z+u4*w;      v: v1*x+v2*y+v3*z+v4*w;


## Verify that ru-sv in the ideal of relations of D

dotsimp(r.u-s.v);

(D28)                                        0
```

Routine A.7. For D with $|\sigma| = 2$, we compute central annihilators for the line modules $M_l = D/Du + Dv$ given in Lemma V.27. This result is used for the proof of Lemma V.28 in §5.4. In particular, we need scalars a_{ij} so that:

- $f_i = a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}w$ for $i = 1, 2$, and;
- $f_1u + f_2v = \alpha w^2 + \beta\hat{g}_2$ for some $[\alpha : \beta] \in \mathbb{P}^1$;

where $\hat{g}_2$ is the degree 2 central element of D not in $k[w^2]$. As mentioned in the proof of Lemma V.28, we assume that $[\alpha : \beta] = [\gamma : 1]$ for $\gamma \in k$.

```
### AFFINE ###

## Define and simplify the expression f1*u + f2*v.

declare(a11,constant,a12,constant,a13,constant,a14,constant,

a21,constant,a22,constant,a23,constant,a24,constant);

f1:a11*x+a12*y+a13*z+a14*w;

f2:a21*x+a22*y+a23*z+a24*w;
```

```
declare(u2,constant,u3,constant,u4,constant,v1,constant,v4,constant);

u1: 1;        v2: u2*v1;       v3: v1*u3;

u: u1*x+u2*y+u3*z+u4*w;

v: v1*x+v2*y+v3*z+v4*w;

dotsimp(f1.u+f2.v);

/R/ (((a22 C u3 - a23 C u2) v1 + a12 C u3 - a13 C u2) (Y . z)

 + ((a22 C u2 - a23 C^2 ) v1 + a12 C u2 - a13 C^2 ) (Y . Y)

 + ((- a23 u3 + a22 C) v1 - a13 u3 + a12 C) (Y . X)

 + ((a21 C u3 - a23 C) v1 + a11 C u3 - a13 C) (X . z)

 + ((- a23 u3 + a21 C u2) v1 - a13 u3 + a11 C u2) (X . Y)

 + ((- a23 C^2  u2 + a21 C) v1 - a13 C^2  u2 + a11 C) (X . X)

 + (a23 C v4 + (- a23 d3 + a24 C) u3 v1 + a13 C u4

    + (- a13 d3 + a14 C) u3) (w . z)

 + (a22 C v4 + (a24 C u2 - a23 C d2) v1 + a12 C u4 + a14 C u2

    - a13 C d2) (w . Y)

 + (a21 C v4 + (- a23 C d1 u2 + a24 C) v1 + a11 C u4

    - a13 C d1 u2 + a14 C) (w . X)

 + (a24 C v4 + (- a23 e3 u3 - a23 C e1 u2 - a23 C e2) v1 +

    a14 C u4 - a13 e3 u3 - a13 C e1 u2 - a13 C e2) (w . w))/C


### MAPLE ###

## We extract the coefficients of the simplified expression

##   f1*u + f2*v. Call such coefficients "annrel**".

annrelyz:=(1/c)*((a22*c*u3 - a23*c*u2)*v1 + a12*c*u3 - a13*c*u2);

annrelyy:=(1/c)*((a22*c*u2 - a23*c^2)*v1 + a12*c*u2 - a13*c^2);
```

```
annrelyx:=(1/c)*((-a23*u3 + a22*c)*v1 - a13*u3 + a12*c);

annrelxz:=(1/c)*((a21*c*u3 - a23*c)*v1 + a11*c*u3 - a13*c);

annrelxy:=(1/c)*((-a23*u3 + a21*c*u2)*v1 - a13*u3 + a11*c*u2);

annrelxx:=(1/c)*((-a23*c^2*u2 + a21*c)*v1 - a13*c^2*u2 + a11*c);

annrelwz:=(1/c)*(a23*c*v4 + (-a23*d3 + a24*c)*u3*v1 + a13*c*u4
 + (- a13*d3 + a14*c)*u3);

annrelwy:=(1/c)*(a22*c*v4 + (a24*c*u2 - a23*c*d2)*v1 + a12*c*u4
 + a14*c*u2 - a13*c*d2);

annrelwx:=(1/c)*(a21*c*v4 + (-a23*c*d1*u2 + a24*c)*v1 + a11*c*u4
 - a13*c*d1*u2 + a14*c);

annrelww:=(1/c)*(a24*c*v4 + (-a23*e3*u3 - a23*c*e1*u2 - a23*c*e2)*v1
 + a14*c*u4 - a13*e3*u3 - a13*c*e1*u2 - a13*c*e2);


## Now considering the coefficients of the central element \hat{g2},
##   we solve for the a(ij).
## Note that to get a solution, we must also solve for u2 and u3.
v:=solve({annrelyz=0,annrelyy=(c^4-c)*d2,annrelyx=(-c^3+1)*d3,
          annrelxz=0,annrelxy=(-c^3+1)*d3,annrelxx=(c^4-c)*d1,
          annrelwz=d3^2-c^2*d1*d2,annrelwy=-c^2*d1*d3 + c^3*d2^2,
          annrelwx=-c^2*d2*d3 + c^3*d1^2},
         [a11,a12,a13,a14,a21,a22,a23,a24,u2,u3]);


## We get one long solution, v.
> nops(v);
                                    1
```

```
## Set the a(ij) equal to the corresponding solutions of v[1][k].

a11:= rhs(v[1][1]);      a12:= rhs(v[1][2]);

a13:= rhs(v[1][3]);      a14:= rhs(v[1][4]);

a21:= rhs(v[1][5]);      a22:= rhs(v[1][6]);

a23:= rhs(v[1][7]);      a24:= rhs(v[1][8]);

u2:=  rhs(v[1][9]);      u3:=  rhs(v[1][10]);

## Now gamma equals the following:

annrelww:=(1/c)*(a24*c*v4 + (-a23*e3*u3 - a23*c*e1*u2 - a23*c*e2)*v1

 + a14*c*u4 - a13*e3*u3 - a13*c*e1*u2 - a13*c*e2);

gamma:= simplify(annrelww);

## This is a long expression, and it is dependent on u4, v1, v4.
```

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