Introductory notes in topology

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1 Topological spaces

A topology on a set \( X \) is a collection \( \tau \) of subsets of \( X \), the open subsets of \( X \) with respect to the topology, such that the empty set \( \emptyset \) and \( X \) itself are open sets, the intersection of finitely many open sets is an open set, and the union of any family of open sets is an open set. For any set \( X \), the indiscrete topology has \( \emptyset, X \) as the only open sets, and the discrete topology is defined by saying that all subsets of \( X \) are open. On the real line \( \mathbb{R} \), a set \( U \subseteq \mathbb{R} \) is open in the standard topology if for every \( x \in U \) there are real numbers \( a, b \) such that \( a < x < b \) and \( (a, b) \subseteq U \), where \( (a, b) \) is the open interval consisting of all real numbers \( r \) such that \( a < r < b \).

Let \( (X, \tau) \) be a topological space. We say that \( X \) satisfies the first separation condition if for every \( x, y \in X \) with \( x \neq y \) there is an open set \( V \subseteq X \) such that \( y \in V \) and \( x \notin V \). To be more precise, \( x \) and \( y \) may be chosen independently of each other, and so the first separation condition actually means that there are a pair of open subsets \( V \) and \( V \) of \( X \) such that \( y \in V \) and \( x \notin V \) as well as \( x \in V \) and \( y \notin V \). We say that \( X \) satisfies the second separation condition, or equivalently that \( X \) is a Hausdorff topological space, if for every \( x, y \in X \) with \( x \neq y \) there are open subsets \( U, V \) of \( X \) such that \( x \in U, y \in V \), and \( U, V \) are disjoint, i.e., \( U \cap V = \emptyset \). For the first homework assignment, discuss the case where \( X \) is a topological space with only finitely many elements which satisfies the first separation condition, and give an example of a topological space that satisfies the first separation condition but not the second.

1.1 Neighborhoods

The definition of a topological space may seem a bit strange at first, and certainly quite abstract. It is sometimes helpful to look at a topological space \( X \) in terms of neighborhoods, where a neighborhood of a point \( x \in X \) can be defined as any open set \( U \subseteq X \) such that \( x \in U \). Alternatively, one may take the neighborhoods of \( x \) in \( X \) to be the subsets of \( X \) that contain an open set containing \( x \). At any rate, the neighborhoods of \( x \) in \( X \) are supposed to contain
all elements of $X$ which are sufficiently close to $x$. The definition of a topology is a convenient way to make this precise, which can then be used for related notions of convergence, continuity, etc. In practice, one might start with a class of neighborhoods of elements of a set $X$, and use this to get a topology on $X$. Specifically, a set $U \subseteq X$ would be defined to be an open set in this situation if for every $x \in U$ there is a neighborhood of $x$ in $X$ which is contained in $U$.

For example, the standard topology on the real line is essentially defined in this way, using open intervals as a basic class of neighborhoods in $\mathbb{R}$.

### 2 Other topologies on $\mathbb{R}$

In addition to the standard topology on the real line $\mathbb{R}$, let us consider a couple of “exotic topologies” $\tau_-, \tau_+$, defined as follows. We say that $U \subseteq \mathbb{R}$ is an open set with respect to the topology $\tau_-$ if for every $x \in U$ there is a real number $a < x$ such that $(a, x] \subseteq U$, where the interval $(a, x]$ consists of the real numbers $t$ which satisfy $a < t \leq x$. Similarly, $U \subseteq \mathbb{R}$ is an open set with respect to the topology $\tau_+$ if for every $x \in U$ there is a real number $b > x$ such that $[x, b) \subseteq U$, where $[x, b)$ consists of the real numbers $t$ which satisfy $x \leq t < b$. One can check that these are topologies on $\mathbb{R}$, and that $U \subseteq \mathbb{R}$ is an open set in the standard topology if and only if it is open with respect to $\tau_-$ and $\tau_+$.

If $(X, \tau)$ is a topological space and $E \subseteq X$, then we say that $E$ is dense in $X$ if for every nonempty open set $U \subseteq X$, $E \cap U \neq \emptyset$. Equivalently, $E$ is dense in $X$ if for every $x \in X$ and every open set $U \subseteq X$ such that $x \in U$, there is a point $y \in E$ which approximates $x$ in the sense that $y \in U$. Observe that $X$ is automatically dense in itself, and if $X$ is equipped with the discrete topology, then $X$ is the only dense set in $X$. One can check that the set $\mathbb{Q}$ of rational numbers is dense in the real line with respect to the standard topology, and also with respect to the topologies $\tau_-, \tau_+$ described in the previous paragraph. This uses the fact that for every pair of real numbers $a, b$ with $a < b$ there is a rational number $r$ such that $a < r < b$.

### 3 Closed sets

Let $(X, \tau)$ be a topological space, and let $E \subseteq X$ and $p \in X$ be given. We say that $p$ is adherent to $E$ in $X$, or equivalently that $p$ is an adherent point of $E$ in $X$, if for every open set $U \subseteq X$ such that $p \in U$ there is an $x \in E$ with $x \in U$. We say that $p$ is a limit point of $E$ in $X$ if for every open set $U \subseteq X$ such that $p \in U$ there is a $y \in E$ with $y \neq p$ and $y \in U$. Every element of $E$ is automatically adherent to $E$, and every limit point of $E$ is automatically adherent to $E$ as well. An adherent point of $E$ in $X$ which is not an element of $E$ is automatically a limit point of $E$, and a limit point of $E$ may or may not be an element of $E$.

The complement of a subset $A$ of $X$ is denoted $X \setminus A$ and defined to be the set of $z \in X$ such that $z \notin A$. Thus the complement of the complement of $A$ is
equal to $A$. The complement of the union of a family of subsets of $X$ is equal to the intersection of the corresponding family of complements of the sets, and similarly the complement of the intersection of a family of subsets of $X$ is equal to the union of the corresponding family of complements of the sets. We say that $A \subseteq X$ is closed if the complement of $A$ in $X$ is an open set. It follows from the definition of a topology that $\emptyset, X$ are closed subsets of $X$, the union of finitely many closed subsets of $X$ is a closed set, and the intersection of any family of closed subsets of $X$ is a closed set.

The closure of $E \subseteq X$ is denoted $\overline{E}$ and defined to be the set of points in $X$ that are adherent to $E$, which is the same as the union of $E$ and the set of limit points of $E$ in $X$. Equivalently, $z \in X$ is not in the closure of $E$ if there is an open set $U \subseteq X$ such that $z \in U$ and $E \cap U = \emptyset$, in which case $U$ is contained in the complement of the closure of $E$ in $X$. It follows that $X \setminus \overline{E}$ is equal to the union of the open subsets of $X$ contained in $X \setminus E$, so that $X \setminus \overline{E}$ is also an open set in $X$, which implies that $\overline{E}$ is a closed set in $X$. If $E$ is a closed subset of $X$, then it is easy to see that $\overline{E} = E$, because every element of the complement of $E$ is contained in the open set $X \setminus E$ which does not intersect $E$ by definition, and hence is not adherent to $E$. Note that a subset $E$ of $X$ is dense in $X$ if and only if $\overline{E} = X$.

### 3.1 Interiors of sets

Let $X$ be a topological space. The interior of a set $A \subseteq X$ is denoted $\text{Int} A$, and defined to be the set of $x \in A$ for which there is an open set $U \subseteq X$ such that $x \in U$ and $U \subseteq A$. If $U \subseteq X$ is any open set such that $U \subseteq A$, then every element of $U$ is contained in $A$, so that $U \subseteq \text{Int} A$.

Thus $\text{Int} A$ may be described equivalently as the union of all of the open subsets of $X$ that are contained in $A$. In particular, it follows that $\text{Int} A$ is automatically an open subset of $X$ too. It is easy to check that the complement of the interior of $A \subseteq X$ in $X$ is equal to the closure of the complement of $A$ in $X$, which is to say that

$$X \setminus \text{Int} A = \overline{X \setminus A}.$$

This is essentially the same as saying that

$$X \setminus \overline{E} = \text{Int}(X \setminus E)$$

for every $E \subseteq X$, which was also mentioned earlier, in effect.

If $A_1, A_2$ are subsets of $X$, then

$$\text{Int}(A_1 \cap A_2) = (\text{Int} A_1) \cap (\text{Int} A_2).$$

This follows from the definition of the interior of a set. Similarly,

$$(\text{Int} A_1) \cup (\text{Int} A_2) \subseteq \text{Int}(A_1 \cup A_2).$$
Thus, 
\[ E_1 \cup E_2 = E_1' \cup E_2' \]
and 
\[ E_1 \cap E_2 \subseteq E_1' \cap E_2' \]
for every \( E_1, E_2 \subseteq X \).

4 Metric spaces

By a metric space we mean a set \( M \) together with a function \( d(x, y) \) defined for \( x, y \in M \) such that (i) \( d(x, y) \) is a nonnegative real number for every \( x, y \in M \) which is equal to 0 if and only if \( x = y \), (ii) \( d(x, y) = d(y, x) \) for every \( x, y, z \in M \), and (iii) the triangle inequality holds, which is to say that 
\[ d(x, z) \leq d(x, y) + d(y, z) \]
for every \( x, y, z \in M \). In this event we say that \( d(x, y) \) defines a metric on \( M \).

For every \( x \in M \) and positive real number \( r \), we put 
\[ B(x, r) = \{ y \in M : d(x, y) < r \} \]
the open ball centered at \( x \) with radius \( r \) in \( M \). We say that \( U \subseteq M \) is an open set if for every \( x \in U \) there is an \( r > 0 \) such that \( B(x, r) \subseteq U \). One can check that this defines a topology on \( M \). Specifically, if \( U_1, \ldots, U_n \) are open subsets of \( M \) in this sense and \( x \) is an element of the intersection \( U_1 \cap \cdots \cap U_n \), then for each \( i = 1, \ldots, n \), \( x \) is an element of \( U_i \), and there is an \( r_i > 0 \) such that \( B(x, r_i) \subseteq U_i \). If \( r \) is equal to the minimum of \( r_1, \ldots, r_n \), then \( r > 0 \) and \( B(x, r) \) is contained in \( U_1 \cap \cdots \cap U_n \). This shows that \( U_1 \cap \cdots \cap U_n \) is an open set in \( M \), and the fact that arbitrary unions of open subsets of \( M \) are open sets is even simpler. For every \( x \in M \) and \( r > 0 \), one can check that the open ball \( B(x, r) \) is an open set, because for every \( y \in B(x, r) \) we have that 
\[ B(y, t) \subseteq B(x, r) \]
with \( t = r - d(x, y) > 0 \), by the triangle inequality.

On any set \( M \), the discrete metric is defined by putting \( d(x, y) \) equal to 0 when \( x = y \) and to 1 when \( x \neq y \). One can check that this is a metric, and that the corresponding topology on \( M \) is the discrete topology.

4.1 Other metrics

Let \( (M, d(x, y)) \) be a metric space. For every positive real number \( \rho \), \( \rho d(x, y) \) defines a metric on \( M \) which determines the same topology on \( M \) as \( d(x, y) \) does, i.e., the same class of open subsets of \( M \). To be more precise, the open ball in \( M \) centered at a point \( w \in M \) and with radius \( t > 0 \) with respect to the initial metric \( d(x, y) \) is the same as the open ball in \( M \) centered at \( w \) with radius \( \rho t \) with respect to the new metric \( \rho d(x, y) \). Similarly, for every positive real
number $\tau$, $d_\tau(x, y) = \min(d(x, y), \tau)$ defines a metric on $M$ which determines the same topology as $d(x, y)$. In order to check that $d_\tau(x, y)$ satisfied the triangle inequality, observe that

$$\min(a + b, \tau) \leq \min(a, \tau) + \min(b, \tau)$$

for every $a, b \geq 0$. The open ball in $M$ centered at a point $z \in M$ with and with radius $r > 0$ with respect to $d(x, y)$ is the same as the open ball in $M$ centered at $z$ with radius $r$ with respect to $d_\tau(x, y)$ when $r \leq \tau$. When $r > \tau$, any open ball in $M$ with radius $r$ with respect to $d_\tau(x, y)$ contains every element of $M$. However, the equivalence of balls with small radius is sufficient to imply that the two metrics determine the same class of open subsets of $M$, i.e., the same topology. As another example along these lines, consider $d'(x, y) = \sqrt{d(x, y)}$.

For any real numbers $a, b \geq 0$,

$$a^2 + b^2 \leq a^2 + 2ab + b^2 = (a + b)^2,$$

and one can use this to show that $d'(x, y)$ satisfies the triangle inequality and hence is a metric on $M$. In this case, the open ball centered at a point $p \in M$ and with radius $r > 0$ with respect to $d(x, y)$ is the same as the open ball centered at $p$ with radius $\sqrt{r}$ with respect to $d'(x, y)$, and this implies that the associated topologies are the same.

5 The real numbers

If $x$ is a real number, then the absolute value $|x|$ of $x$ is equal to $x$ when $x \geq 0$ and to $-x$ when $x \leq 0$. Hence $|x| \geq 0$ for every $x \in \mathbb{R}$, and $|x| = 0$ if and only if $x = 0$. One can check that

$$|x + y| \leq |x| + |y|$$

for every $x, y \in \mathbb{R}$, and therefore that $|x - y|$ defines a metric on $\mathbb{R}$, known as the standard metric. The topology determined by the standard metric is the standard topology. To be more precise, the open ball in $\mathbb{R}$ centered at $x \in \mathbb{R}$ with radius $r > 0$ is the same as the open interval $(x - r, x + r)$.

If $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$, then we say that $b$ is an upper bound for $A$ if $a \leq b$ for every $a \in A$. We say that $c \in \mathbb{R}$ is the least upper bound or supremum of $A$, denoted $\sup A$, if $c$ is an upper bound for $A$ and if $c \leq b$ for every upper bound $b$ of $A$. It follows directly from the definition that the supremum of $A$ is unique if it exists, i.e., if $c_1, c_2 \in \mathbb{R}$ both satisfy the requirements of suprema of $A$, then $c_1 \leq c_2, c_2 \leq c_1$, and thus $c_1 = c_2$. The completeness property of the real line states that every nonempty set of real numbers which has an upper bound has a supremum. The rational numbers are not complete in this sense, because the set of $t \in \mathbb{Q}$ such that $t \geq 0$ and $t^2 \leq 2$ is a nonempty subset of $\mathbb{Q}$ with an upper bound that does not have a supremum in $\mathbb{Q}$, and in fact the supremum in the real line is equal to $\sqrt{2}$. 

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Similarly, $w \in \mathbb{R}$ is a lower bound for $B \subseteq \mathbb{R}$ if $w \leq x$ for every $x \in B$, and $z \in \mathbb{R}$ is the greatest lower bound or infimum of $B$, denoted $\inf B$, if $z$ is a lower bound of $B$ and every lower bound $w \in \mathbb{R}$ of $B$ satisfies $w \leq z$. As in the previous situation, the infimum is automatically unique if it exists. If $B$ is a nonempty set of real numbers with a lower bound, then the completeness property for the real numbers implies that $B$ has an infimum, which can be obtained as the supremum of the set of lower bounds of $B$. Alternatively, the infimum of $B$ is equal to the negative of the supremum of 

$$-B = \{ x \in \mathbb{R} : -x \in B \}.$$ 

### 5.1 Additional properties

The completeness property of the real numbers can be used to show that every $a \in \mathbb{R}$ with $a > 0$ has a positive square root. Specifically, let $A$ be the set of $r \in \mathbb{R}$ such that $r \geq 0$ and $r^2 \leq a$. Clearly $0 \in A$, and thus $A \neq \emptyset$. If $a \leq 1$, then $r \leq 1$ for every $r \in A$, while if $a \geq 1$, then $r \leq a$ for every $a \in A$. Hence $A$ is a nonempty set of real numbers with an upper bound, and thus there is a $b \in \mathbb{R}$ which is the least upper bound of $A$. If $b^2 < a$, then one can show that there is an $\epsilon > 0$ such that $(b+\epsilon)^2 = b^2 + 2b\epsilon + \epsilon^2 < a$. Similarly, if $b^2 > a$, then one can show that there is a $\delta > 0$, $\delta < b$, such that $(b-\delta)^2 = b^2 - 2b\delta + \delta^2 > a$. In both cases one can get a contradiction with the fact that $b = \sup A$, and it follows that $b^2 = a$. It is easy to show directly that a positive square root of $a$ is unique.

If $r, t \in \mathbb{R}$ and $r \leq t$, then the closed interval $[r, t]$ consists of the $x \in \mathbb{R}$ such that $r \leq x \leq t$. Suppose that $I_l = [r_l, t_l]$, $l = 1, 2, \ldots$, is a sequence of closed intervals in the real line such that $I_l \subseteq I_{l+1}$ for every $l \geq 1$. Equivalently, $r_l \leq r_{l+1} \leq t_{l+1} \leq t_l$ for every $l \geq 1$, which implies that $r_l \leq r_n \leq t_n \leq t_l$ when $1 \leq l \leq n$. Using completeness one can show that there is an $x \in \mathbb{R}$ such that $x \in I_l$ for every $l \geq 1$. To be more precise, the intersection $\bigcap_{l=1}^{\infty} I_l$ of all of the $I_l$’s is the same as the closed interval $[r, t]$, where $r$ is the supremum of the set of $r_l$’s and $t$ is the infimum of the set of $t_l$’s, $l \geq 1$.

### 5.2 Diameters of bounded subsets of metric spaces

Let $(M, d(x, y))$ be a metric space. A set $E \subseteq M$ is said to be bounded if there is a point $p \in M$ and a positive real number $r$ such that $d(x, p) \leq r$ for every $x \in E$. In this case, $d(x, q) \leq d(p, q) + r$ for every $q \in M$ and $x \in E$, by the triangle inequality. Equivalently, $E \subseteq M$ is bounded if the set of real numbers of the form $d(x, y)$, $x, y \in E$, is bounded. If $E \subseteq M$ is bounded and nonempty, then the diameter of $E$ is denoted $\text{diam} E$ and defined by 

$$\text{diam} E = \sup \{ d(x, y) : x, y \in E \}.$$ 

Suppose that $E_1$, $E_2$ are nonempty subsets of $M$ such that $E_1 \subseteq E_2$. If $E_2$ is bounded, then $E_1$ is bounded too, and 

$$\text{diam} E_1 \leq \text{diam} E_2.$$
If $E \subseteq M$ is bounded and nonempty, then the closure $\overline{E}$ of $E$ is bounded, and

$$\text{diam } \overline{E} = \text{diam } E.$$  

For if $x, y \in \overline{E}$ and $\epsilon > 0$ is arbitrary, then there are $x', y' \in E$ such that $d(x, x')$, $d(y, y') < \epsilon$. Hence $d(x, y) \leq d(x', y') + 2\epsilon$, by the triangle inequality, which implies that $d(x, y) \leq \text{diam } E + 2\epsilon$. Because this holds for every $x, y \in \overline{E}$, we may conclude that $\overline{E}$ is bounded and $\text{diam } \overline{E} \leq \text{diam } E + 2\epsilon$, which implies in turn that $\text{diam } \overline{E} \leq \text{diam } E$ since $\epsilon > 0$ is arbitrary. The opposite inequality holds automatically, and therefore the diameters of $E$ and $\overline{E}$ are equal.

6 The extended real numbers

By definition, the extended real numbers consist of the real numbers together with two additional elements, $-\infty$ and $+\infty$, such that

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$. The notions of upper and lower bounds make sense for sets of extended real numbers, with $+\infty$ as an upper bound and $-\infty$ as a lower bound for any set of extended real numbers. Using the completeness property for the real numbers, one can check that every nonempty set of extended real numbers has a unique supremum and infimum, defined in the same way as for sets of real numbers. In particular, if $E$ is a nonempty set of real numbers which does not have an upper bound in $\mathbb{R}$, then the supremum of $E$ is equal to $+\infty$ in the extended real numbers. Similarly, the infimum of a nonempty set of real numbers without a finite lower bound is equal to $-\infty$ in the extended real numbers.

A set $U$ of extended real numbers is an open set if for every $x \in U$ either $x \in \mathbb{R}$ and there are real numbers $a, b$ such that $a < x < b$ and $(a, b) \subseteq U$, or $x = -\infty$ and there is a real number $b$ such that the set $[-\infty, b)$ of extended real numbers $y < b$ is contained in $U$, or $x = +\infty$ and there is a real number $a$ such that the set $(a, +\infty]$ of extended real numbers $z > a$ is contained in $U$. This defines a topology on the set of extended real numbers, for which the real line $\mathbb{R}$ is a dense open set. This topology agrees with the one on $\mathbb{R}$ in the sense that $E \subseteq \mathbb{R}$ is open in $\mathbb{R}$ with the standard topology if and only if $E$ is open in the extended real numbers. If $E \subseteq \mathbb{R}$, then $+\infty$ is a limit point of $E$ in the extended real numbers if and only if $E$ does not have an upper bound in $\mathbb{R}$, and similarly $-\infty$ is a limit point of $E$ in the extended real numbers if and only if $E$ does not have a lower bound in $\mathbb{R}$.

7 Relatively open sets

Suppose that $(X, \tau)$ is a topological space and that $Y \subseteq X$. There is a natural topology on $Y$ induced by the one on $X$, in which $E \subseteq Y$ is open in $Y$ when there is an open set $U \subseteq X$ such that $E = U \cap Y$. In this event we say that $E$
is relatively open in $Y$. If $Y$ happens to be an open set in $X$, then $E \subseteq Y$ is relatively open in $Y$ if and only if $E$ is an open set in $X$, but otherwise relatively open subsets of $Y$ may not be open in $X$. Following the general definition of closed subsets of a topological space, $E \subseteq Y$ is relatively closed in $Y$ if $Y \setminus E$ is relatively open in $Y$. Hence $E$ is relatively closed in $Y$ when $Y \setminus E = U \cap Y$ for some open set $U$ in $X$, which implies that $E = (X \setminus U) \cap Y$ and $X \setminus U$ is closed in $X$. Conversely, if $E = A \cap Y$ for some closed set $A$ in $X$, then $Y \setminus E = (X \setminus A) \cap Y$ and $X \setminus A$ is open in $X$, and $E$ is relatively closed in $Y$. In particular, if $Y$ is closed in $X$, then $E \subseteq Y$ is relatively closed in $Y$ if and only if $E$ is closed in $X$. For any $Y$, the relative closure of $E \subseteq Y$, which is the closure of $E$ in $Y$ as a topological space itself, is equal to the intersection of $Y$ with the closure of $E$ in $X$.

If $(M, d(x, y))$ is a metric space and $Y \subseteq M$, then the restriction of $d(x, y)$ to $x, y \in Y$ defines a metric on $Y$. The topology on $Y$, associated to the restriction of the metric to $Y$ is the same as the topology on $Y$ induced from the topology on $X$ determined by the metric. Indeed, if $U \subseteq X$ is open in $X$, then $U \cap Y$ is open in $Y$ with respect to the restriction of the metric to $Y$. Conversely, the open ball in $Y$ with center $y \in Y$ and radius $r > 0$ is automatically equal to the intersection of $Y$ with the open ball in $X$ with center $y$ and radius $r$. Therefore, every open ball in $Y$ is relatively open with respect to the topology induced from the one on $X$. If $E \subseteq Y$ is an open set with respect to the topology determined by the restriction of the metric to $Y$, then $E$ is equal to a union of open balls in $Y$, the open balls in $Y$ that are contained in $E$. Each of these open balls is the intersection of $Y$ with an open ball in $X$, and hence their union is the same as the intersection of $Y$ with the union of the corresponding open balls in $X$, an open set in $X$.

7.1 Additional remarks

Let $X$ be a topological space. If $U \subseteq X$ is an open set, then for every $Y \subseteq X$, $U \cap Y$ is relatively open in $Y$. Conversely, if $U \subseteq X$ and $U \cap Y$ is relatively open in $Y$ for every $Y \subseteq X$, then $U$ is an open set in $X$, since we can take $Y = X$. Similarly, $E \subseteq X$ is a closed set if and only if $E \cap Y$ is relatively closed in $Y$ for every $Y \subseteq X$. As a refinement of this type of question, one can look for classes of subsets $Y$ of $X$ such that open and closed subsets of $X$ are characterized by intersections with $Y$ being relatively open or closed in $Y$, respectively, for every $Y$ in the particular class of subsets of $X$. For example, the class of open subsets $Y$ of $X$ has this property. Actually, any class of open subsets of $X$ whose union is equal to $X$ has this property. Any class of subsets of $X$ for which the union of their interiors is equal to $X$ has this property for the same reasons.

8 Convergent sequences

A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of a topological space $X$ converges to $x \in X$ if for every open set $U \subseteq X$ with $x \in U$ there is a positive integer $L$ such that
$x_j \in U$ for every $j \geq L$. If $X$ is equipped with the indiscrete topology, then every sequence $\{x_j\}_{j=1}^\infty$ of elements of $X$ converges to every $x \in X$. If $X$ is equipped with the discrete topology and $\{x_j\}_{j=1}^\infty$ is a sequence of elements of $X$ that converges to $x \in X$, then there is an $L \geq 1$ such that $x_j = x$ for every $j \geq L$, because $\{x\}$ is an open set in $X$. If $X$ is any topological space, and if there is an $L \geq 1$ such that $x_j = x$ for every $j \geq L$, then $\{x_j\}_{j=1}^\infty$ converges to $x$. In the real line, the sequence $\{x_j\}_{j=1}^\infty$ defined by $x_j = 1/j$ for each $j \geq 1$ converges to 0 with respect to the standard topology.

Let $X$ be a topological space, let $x$ be an element of $X$, and consider the sequence $\{x_j\}_{j=1}^\infty$ such that $x_j = x$ for every $j$. This sequence automatically converges to $x$ in $X$, as in the previous paragraph. If $X$ satisfies the first separation condition, $y \in X$, and $y \neq x$, then there is an open set $V \subseteq X$ such that $y \in V$ and $x \notin V$, and it follows that $\{x_j\}_{j=1}^\infty$ does not converge to $y$ in $X$. In the other direction, if $y \in X$, and if every open set $V \subseteq X$ with $y \in V$ also contains $x$, then $\{x_j\}_{j=1}^\infty$ converges to $y$.

Suppose that $X$ is a topological space, $E \subseteq X$, and $\{x_j\}_{j=1}^\infty$ is a sequence of elements of $E$ which converges to $x \in X$. Under these conditions, $x$ is adherent to $E$, and hence an element of the closure $\overline{E}$ of $E$, and an element of $E$ if $E$ is a closed set. If $x_j \neq x$ for every $j$, then $x$ is a limit point of $E$. Note that the first separation condition is equivalent to the statement that subsets of $X$ with exactly one element are closed. More precisely, for each $x \in X$, the first separation condition implies that $X \setminus \{x\}$ can be expressed as the union of a family of open subsets of $X$, so that $X \setminus \{x\}$ is an open set in $X$ too.

### 8.1 Monotone sequences

A sequence $\{x_l\}_{l=1}^\infty$ of real numbers is said to be monotone increasing if $x_l \leq x_n$ for every $n \geq l \geq 1$. Similarly, a sequence $\{y_l\}_{l=1}^\infty$ of real numbers is said to be monotone decreasing if $y_l \leq y_n$ when $n \geq l \geq 1$. Every monotone increasing or decreasing sequence of real numbers which is bounded in the sense that the terms of the sequence are contained in a bounded set in $\mathbb{R}$ converges. Specifically, one can check that a monotone increasing sequence of real numbers which is bounded from above converges to the supremum of the set of terms in the sequence, and a monotone decreasing sequence of real numbers which is bounded from below converges to the infimum of the terms in the sequence. In particular, this uses the completeness property of the real numbers.

Now consider the real line equipped with the exotic topologies $\tau_+, \tau_-,$ where $U \subseteq \mathbb{R}$ is an open set with respect to $\tau_+$ if and only if for every $x \in U$ there is a $t \in \mathbb{R}$ with $t > x$ such that $[x,t] \subseteq U$, and $V \subseteq \mathbb{R}$ is an open set with respect to $\tau_-$ if and only if for every $y \in V$ there is an $r \in \mathbb{R}$ with $r < y$ such that $(r,y] \subseteq V$. A monotone increasing sequence of real numbers which is bounded from above converges to the supremum of the terms in the sequence with respect to the topology $\tau_+$, but a monotone decreasing sequence converges with respect to $\tau_-$ if and only if it is eventually constant. In the same way, a monotone decreasing sequence of real numbers which is bounded from below converges to the infimum of the terms in the sequence with respect to the topology $\tau_+$, but
a monotone increasing sequence converges with respect to $\tau_+$ if and only if it is eventually constant. To be more precise, a sequence is said to be eventually constant if all but finitely many terms in the sequence are the same.

8.2 Cauchy sequences

Let $(M,d(x,y))$ be a metric space. A sequence $\{x_l\}_{l=1}^\infty$ of elements of $M$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an $L \geq 1$ such that

$$d(x_l,x_n) < \epsilon \quad \text{for every } l,n \geq L.$$  

If $\{x_l\}_{l=1}^\infty$ converges to a point $x \in M$, then $\{x_l\}_{l=1}^\infty$ is a Cauchy sequence, because for every $\epsilon > 0$ there is an $L$ such that

$$d(x_l,x) < \frac{\epsilon}{2} \quad \text{when } l \geq L,$$

and hence

$$d(x_l,x_n) \leq d(x_l,x) + d(x_n,x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

when $l,n \geq L$. A metric space in which every Cauchy sequence converges is said to be complete.

One can use the completeness property of the real numbers in terms of ordering to show that the real line equipped with the standard metric $|x-y|$ is complete as a metric space. For suppose that $\{x_l\}_{l=1}^\infty$ is a Cauchy sequence of real numbers, and consider $E_l = \{x_n : n \geq l\}$ for each $l$. Because $\{x_l\}_{l=1}^\infty$ is a Cauchy sequence, $E_l$ is a bounded set of real numbers for every $l \geq 1$, and hence has an infimum $r_l$ and a supremum $t_l$. The fact that $\{x_l\}_{l=1}^\infty$ is a Cauchy sequence implies that $t_l - r_l$ converges to 0 as a sequence of real numbers. One can check that the supremum of the $r_l$’s is equal to the infimum of the $t_l$’s, and that $\{x_l\}_{l=1}^\infty$ converges to their common value.

9 The local countability condition

A topological space $X$ satisfies the local countability condition at a point $p \in X$ if there is a sequence $\{U_l(p)\}_{l=1}^\infty$ of open subsets of $X$ such that $p \in U_l(p)$ and $U_{l+1}(p) \subseteq U_l(p)$ for every $\ell \geq 1$, and for every open set $U \subseteq X$ with $p \in U$ there is an $\ell \geq 1$ such that $U_l(p) \subseteq U$. It is easy to arrange for the $U_l(p)$’s to be decreasing, by replacing $U_l(p)$ with $U_1(p) \cap \cdots \cap U_l(p)$ if necessary, and this ensures that $U_l(p) \subseteq U$ implies $U_k(p) \subseteq U$ when $k \geq l$. The topology of a metric space satisfies the local countability condition at every point, because one can take $U_\ell(p)$ to be the open ball centered at $p$ with radius $1/\ell$. The real line equipped with the topology $\tau_+$, in which $U \subseteq \mathbb{R}$ is open if and only if for every $p \in U$ there is a $t \in \mathbb{R}$ with $t > p$ such that $[p,t) \subseteq U$, also satisfies the local countability condition at every point, because one can take $U_\ell(p) = [p,p+1/\ell)$. Similarly, the real line equipped with the topology $\tau_-$, in which $U \subseteq \mathbb{R}$ is open if and only if for every $p \in U$ there is an $r \in \mathbb{R}$, $r < t$, such that $(r,t) \subseteq U$, satisfies the local countability condition at every point, with $U_\ell(p) = (p-1/\ell,p]$.  

Suppose that $X$ is a topological space that satisfies the local countability condition at a point $p \in X$, and let $\{U_\ell(p)\}^\infty_{\ell=1}$ be a sequence of open sets as in the previous paragraph. If $p$ is adherent to $E \subseteq X$, then for every $\ell \geq 1$, there is an $x_\ell \in E$ such that $x_\ell \in U_\ell(p)$. It is easy to see that $\{x_\ell\}^\infty_{\ell=1}$ converges to $p$ in $X$ under these conditions. If $p$ is a limit point of $E$, then one can choose $x_\ell$ such that $x_\ell \neq p$ for each $\ell \geq 1$ too. Remember that there are analogous statements in the other direction that work in any topological space.

9.1 Subsequences

Let $\{x_i\}^\infty_{i=1}$ be a sequence of elements of some set $X$. A subsequence of $\{x_i\}^\infty_{i=1}$ is a sequence of the form $\{x_{i_n}\}^\infty_{n=1}$, where $i_1 < i_2 < \cdots$ is a strictly increasing sequence of positive integers. Suppose now that $X$ is a topological space. Let $E_1$ be the set of $x \in X$ for which there is a subsequence $\{x_{i_n}\}^\infty_{n=1}$ of $\{x_i\}^\infty_{i=1}$ such that $\{x_{i_n}\}^\infty_{n=1}$ converges to $x$ in $X$. Let $E_2$ be the set of $x \in X$ with the property that for every open set $U \subseteq X$ with $x \in U$ there are infinitely many positive integers $l$ such that $x_l \in U$. It follows directly from the definitions that $E_1 \subseteq E_2$. Conversely, if $X$ satisfies the local countability condition at each point, then one can check that $E_2 \subseteq E_1$ and hence $E_1 = E_2$. In any topological space $X$, it is easy to see that $E_2$ is a closed set. Therefore, if $X$ satisfies the local countability condition at each point, then $E_1$ is a closed set in $X$ too.

9.2 Sequentially closed sets

Let $(X, \tau)$ be a topological space. A set $E \subseteq X$ is said to be sequentially closed if for every sequence $\{x_i\}^\infty_{i=1}$ of elements of $E$ and every $x \in X$ such that $\{x_i\}^\infty_{i=1}$ converges to $x$ in $X$ we have that $x \in E$. Closed subsets of $X$ in the usual sense are automatically sequentially closed. Conversely, sequentially closed subsets of $X$ are closed in the usual sense when $X$ satisfies the local countability condition at each point. More generally, if $Y \subseteq X$ satisfies the local countability condition at each of its elements with respect to the topology induced by $\tau$ on $X$ and if $E$ is sequentially closed in $X$, then $E \cap Y$ is relatively closed in $Y$.

It is easy to see that the intersection of any family of sequentially closed subsets of $X$ is sequentially closed. If $E_1, E_2$ are sequentially closed subsets of $X$, then their union $E_1 \cup E_2$ is sequentially closed, because any sequence of elements of $E_1 \cup E_2$ has a subsequence whose terms are contained in $E_1$ or in $E_2$. Of course a subsequence of a convergent sequence converges to the same limit. It follows that the collection $\widehat{\tau}$ of complements of sequentially closed subsets of $X$ defines a topology on $X$ that contains $\tau$. However, it is not clear that a sequence in $X$ which converges with respect to $\tau$ also converges with respect to $\widehat{\tau}$.
10 Local bases

A local base for the topology of a topological space $X$ at a point $p \in X$ is a collection $\mathcal{B}(p)$ of open subsets of $X$ such that $p \in V$ for every $V \in \mathcal{B}(p)$, and for every open set $U \subseteq X$ with $p \in U$ there is a $V \in \mathcal{B}(p)$ which satisfies $V \subseteq U$. The collection of all open subsets of $X$ that contain $p$ as an element is a local base for the topology at $p$. In a metric space, the open balls centered at $p$ form a local base for the topology at $p$. The exotic topologies $\tau_+,$ $\tau_-$ on the real line are characterized by the local bases $\mathcal{B}_+(p),$ $\mathcal{B}_-(p),$ where $\mathcal{B}_+(p)$ consists of the intervals $(p,t),$ $t > p,$ and $\mathcal{B}_-(p)$ consists of the intervals $(r,p),$ $r < p.$ In any topological space, the topology is determined by choices of local bases at all of the elements of the space. A topological space $X$ satisfies the local countability condition at a point $p \in X$ if and only if there is a local base $\mathcal{B}(p)$ for the topology at $p$ whose elements can be enumerated by a sequence. As noted previously, a sequence of open subsets of $X$ that form a local base for the topology at $p$ using the intersection of the first $n$ open sets in the original sequence for each $n.$

Let $\mathcal{B}(p)$ be a local base for the topology of $X$ at $p,$ and let $E \subseteq X$ be given. It is easy to see that $p$ is adherent to $E$ if and only if for every $U \in \mathcal{B}(p)$ there is a $q \in E$ such that $q \in U$. Similarly, $p$ is a limit point of $E$ if and only if for every $U \in \mathcal{B}(p)$ there is a $q \in E$ such that $q \in U$ and $q \neq p$.

11 Nets

A partial order on a set $A$ is a binary relation $\prec$ such that (i) $x \prec x$ for every $x \in A,$ (ii) $x \prec y$ and $y \prec x$ imply $x = y$ for every $x,y \in A,$ and (iii) $x \prec y$ and $y \prec z$ imply $x \prec z$ for every $x,y,z \in A.$ A partially-ordered set $(A,\prec)$ is a directed system if for every collection $a_1,\ldots,a_n$ of finitely many elements of $A$ there is an $a \in A$ such that $a_i \prec a$ for $i = 1,\ldots,n$. For example, the set of positive integers is a directed system with respect to the usual ordering. If $X$ is a topological space, $p \in X,$ and $\mathcal{B}(p)$ is a local base for the topology of $X$ at $p,$ then $\mathcal{B}(p)$ becomes a directed system with the ordering $\prec$ defined by $U \prec V$ when $V \subseteq U$ for every $U,V \in \mathcal{B}(p).$ The ordering is the reverse of inclusion, because smaller open sets represent greater precision at $p.$ If $U_1,\ldots,U_n \in \mathcal{B}(p),$ then their intersection $U_1 \cap \cdots \cap U_n$ is an open set in $X$ that contains $p,$ and hence there is a $V \in \mathcal{B}(p)$ such that $V \subseteq U_1 \cap \cdots \cap U_n.$ This shows that $\mathcal{B}(p)$ is a directed system with respect to the ordering $\prec$.

If $(A,\prec)$ is a directed system and $X$ is any set, then a net $\{x_a\}_{a \in A}$ indexed by $A$ with values in $X$ assigns to every $a \in A$ an element $x_a$ of $X$. If $A$ is the set of positive integers with the usual ordering, then this is the same as a sequence of elements of $X$. If $X$ is a topological space, $\{x_a\}_{a \in A}$ is a net of elements of $X$ along the directed system $A,$ and $x \in X,$ then $\{x_a\}_{a \in A}$ is said to converge to $x$ if for every open set $U \subseteq X$ there is a $b \in A$ such that $x_a \in U$ for every $a \in A$ which satisfies $b \prec a.$ This is the same as the definition of convergence.
11.1 Sub-limits

Let $X$ be a topological space, let $(A, \prec)$ be a directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of $X$ indexed by $A$. Let $E$ be the set of $x \in X$ with the property that for every $a \in A$ and open set $U \subseteq X$ with $x \in U$ there is a $b \in A$ such that $a \prec b$ and $x_b \in U$. It is easy to see from the definition that $E$ is a closed set in $X$. Fix an $x \in E$. Let $A_x$ be the set of ordered pairs $(a, U)$ where $a \in A$ and $U \subseteq X$ is an open set with $x \in U$. It would be sufficient for the present purposes to use only open sets $U$ in a local base for the topology of $X$ at $x$. If $(a, U), (b, V) \in A_x$, then put $(a, U) \prec_x (b, V)$ when $a \prec b$ and $V \subseteq U$.

One can check that this defines a partial ordering on $A_x$ which makes $A_x$ into a directed system. For $(a, U) \in A_x$, let $x_{(a,U)}$ be an element of $X$ of the form $x_b$, where $b \in A$, $a \prec b$, and $x_b \in U$. By construction, $\{x_{(a,U)}\}_{(a,U) \in A_x}$ is a net of elements of $X$ indexed by $A_x$ that converges to $x$.

12 Uniqueness of limits

Let $X$ be a Hausdorff topological space, let $(A, \prec)$ be a directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of $X$ which converges to $x, y \in X$. If $x \neq y$, then the Hausdorff property implies that there are open subsets $U, V$ of $X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Because of convergence, there are $b, b' \in A$ such that $x_a \in U$ when $b \prec a$ and $x_a \in V$ when $b' \prec a$. If $c \in A$ and $b, b' \prec c$, then $x_a \in U \cap V$ when $c \prec a$, a contradiction. This shows that the limit of a convergent net in a Hausdorff space is unique, and in particular the limit of a convergent sequence is unique.

Conversely, suppose that $X$ is a topological space which is not Hausdorff. Hence there are $x, y \in X$ such that $x \neq y$ and for every pair of open subsets $U$, $V$ of $X$ with $x \in U$ and $y \in V$, we have that $U \cap V \neq \emptyset$. Let $A$ be the set of ordered pairs $(U, V)$ of open subsets of $X$ with $x \in U$ and $y \in V$. There is a natural partial ordering $\prec$ on $A$, where $(U, V) \prec (U', V')$ for $(U, V), (U', V') \in A$ when $U' \subseteq U$ and $V' \subseteq V$. If $\{U_1, V_1\}, \ldots, (U_n, V_n)$ are finitely many elements of $A$, and if we put $U = U_1 \cap \cdots \cap U_n$, $V = V_1 \cap \cdots \cap V_n$, then $(U, V) \in A$ and $(U_i, V_i) \prec (U, V)$ for $i = 1, \ldots, n$. This shows that $(A, \prec)$ is a directed system.

For each $(U, V) \in A$, let $x_{(U,V)}$ be an element of $U \cap V$. This defines a net along $A$ with values in $X$. By construction, this net converges to $x$ and to $y$. It follows that a topological space is Hausdorff if every convergent net in the space has a unique limit.
13 Regularity

Let \((M, d(x, y))\) be a metric space, and let \(x, y\) be elements of \(M\), with \(x \neq y\). If \(r = d(x, y)\), then \(B(y, r)\) is an open set in \(M\) that contains \(y\) but not \(x\), and hence the topology on \(M\) determined by the metric satisfies the first separation condition. If \(t = d(x, y)/2\), then \(B(x, t), B(y, t)\) are disjoint open sets in \(M\) which contain \(x, y\), respectively, and it follows that \(M\) is a Hausdorff topological space. For every \(x \in M\) and \(r \geq 0\), let \(V_r(x)\) be the set of \(w \in M\) such that \(d(w, x) > r\). If \(z \in V_r(x)\) and \(t = d(z, x) - r\), then \(t\) is a positive real number, and one can use the triangle inequality to check that \(B(z, t)\) is contained in \(V_r(x)\). It follows that \(V_r(x)\) is an open set in \(M\). For \(x \in M\) and \(r \geq 0\), the closed ball with center \(x\) and radius \(r\) is denoted \(\overline{B}(x, r)\) and defined to be the set of \(w \in M\) with \(d(w, x) \leq r\). This is the same as the complement of \(V_r(x)\) in \(M\), and hence \(\overline{B}(x, r)\) is a closed set in \(M\). One can also show that \(\overline{B}(x, r)\) is a closed set by verifying directly that every limit point of \(\overline{B}(x, r)\) in \(M\) is an element of \(\overline{B}(x, r)\).

A topological space \(X\) satisfies the third separation condition, or is regular, if it satisfies the first separation condition, and if for every \(x \in X\) and closed set \(E \subseteq X\) such that \(x \notin E\) there are open subsets \(U, V\) of \(X\) such that \(x \in U, E \subseteq V\), and \(U \cap V = \emptyset\). The first separation condition is equivalent to the statement that for every \(y \in X\), the set consisting of \(y\) and no other elements is closed in \(X\), and therefore a regular topological space is Hausdorff because one can take \(E = \{y\}\) when \(y \neq x\). Suppose that \((M, d(x, y))\) is a metric space, \(x \in M\), and \(E \subseteq M\) is a closed set such that \(x \in M \setminus E\). Because \(M \setminus E\) is an open set, there is \(r > 0\) such that \(B(x, r) \subseteq M \setminus E\). It follows that \(B(x, r/2)\), \(V_{r/2}(x)\) are disjoint open subsets of \(M\) that contain \(x, E\), respectively, and thus \(M\) is a regular topological space.

13.1 Subspaces

Let \(X\) be a topological space, and let \(Y\) be a subset of \(X\), equipped with the topology induced by the one on \(X\). If \(X\) satisfies the first separation condition, then it is easy to see that \(Y\) does too. Namely, if \(p\) and \(q\) are distinct elements of \(Y\), then \(p\) and \(q\) are also distinct elements of \(X\), and hence there is an open set \(V\) in \(X\) such that \(q \in V\) and \(p \notin V\). This implies that \(V \cap Y\) is a relatively open set in \(Y\) that contains \(q\) and not \(p\), as desired. Similarly, if \(X\) is Hausdorff, then \(Y\) is Hausdorff too. Suppose now that \(X\) is regular, and let us check that \(Y\) is regular as well. Of course, \(X\) satisfies the first separation condition in this case, and hence \(Y\) does too. Let \(x\) be an element of \(Y\), and let \(E\) be a relatively closed subset of \(Y\) that does not contain \(x\). Thus there is a closed set \(E_1\) in \(X\) such that \(E = E_1 \cap Y\), and \(x \notin E_1\), since \(x \in Y \setminus E\). This implies that there are open sets \(U_1, V_1\) in \(X\) such that \(x \in U_1, E_1 \subseteq V_1\), and \(U_1 \cap V_1 = \emptyset\), because \(X\) is regular. Therefore \(U_1 \cap Y\) and \(V_1 \cap Y\) are relatively open subsets of \(Y\) such that \(x \in U_1 \cap Y, E \subseteq V_1 \cap Y\), and \((U_1 \cap Y) \cap (V_1 \cap Y) = \emptyset\), as desired.
14 An example

Let $V$ be the space of real-valued functions on the set $\mathbb{Z}_+$ of positive integers. This is basically the same as the space of sequences of real numbers, but for the sake of notational simplicity it is convenient to express the elements of $V$ as functions. If $\rho$ is a positive real-valued function on $\mathbb{Z}_+$ and $f \in V$, then put

$$N_{\rho}(f) = \{g \in V : |f(l) - g(l)| < \rho(l) \text{ for every } l \in \mathbb{Z}_+\},$$

the neighborhood of $f$ associated to $\rho$ in $V$. Let us say that $U \subseteq V$ is an open set if for every $f \in U$ there is a positive real-valued function $\rho$ on $\mathbb{Z}_+$ such that $N_{\rho}(f) \subseteq U$. Clearly the empty set and $V$ are open subsets of $V$ in this sense. Suppose that $U_1, \ldots, U_n$ are open subsets of $V$, and that $f \in U_1 \cap \cdots \cap U_n$. For each $i = 1, \ldots, n$, there is a positive real-valued function $\rho_i$ on $\mathbb{Z}_+$ such that $N_{\rho_i}(f) \subseteq U_i$. If

$$\rho(l) = \min(\rho_1(l), \ldots, \rho_n(l))$$

for $l \in \mathbb{Z}_+$, then $\rho$ is a positive real-valued function on $\mathbb{Z}_+$, and

$$N_{\rho}(f) = N_{\rho_1}(f) \cap \cdots \cap N_{\rho_n}(f) \subseteq U_1 \cap \cdots \cap U_n.$$

Hence $U_1 \cap \cdots \cap U_n$ is an open set in $V$. It is easy to see from the definition that a union of open subsets of $V$ is an open set, and therefore that we have a topology on $V$.

If $f \in V$, $\rho$ is a positive real-valued function on $\mathbb{Z}_+$, and $g \in N_{\rho}(f)$, then

$$\rho'(l) = \rho(l) - |f(l) - g(l)| > 0$$

for every $l \in \mathbb{Z}_+$ and $N_{\rho'}(g) \subseteq N_{\rho}(f)$, which implies that $N_{\rho}(f)$ is an open set in $V$. If $f_1, f_2$ are distinct elements of $V$, which means that $f_1(l) \neq f_2(l)$ for some $l$, then one can choose a positive real-valued function $\rho$ on $\mathbb{Z}_+$ such that $2 \rho(l) < |f_1(l) - f_2(l)|$ for some $l$, and hence $N_{\rho}(f_1) \cap N_{\rho}(f_2) = \emptyset$. Thus $V$ is Hausdorff, and one can also show that $V$ is regular. However, $V$ does not satisfy the local countability condition at any point. For suppose that $f \in V$ and that $U_1, U_2, \ldots$ is a sequence of open sets in $V$ that contain $f$ and form a local base for the topology of $V$ at $f$. Let $\rho_1, \rho_2, \ldots$ be a sequence of positive real-valued functions on $\mathbb{Z}_+$ such that $N_{\rho_i}(f) \subseteq U_i$ for each $i$. Put

$$\rho(n) = \frac{\min(\rho_1(n), \ldots, \rho_n(n))}{n}.$$

Thus $\rho$ is a positive real-valued function on $\mathbb{Z}_+$, $\rho(l) < \rho_i(l)$ when $l \geq i$, and $N_{\rho}(f)$ does not contain $N_{\rho_i}(f) \subseteq U_i$ for any $i$, a contradiction.

If $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $V$, $f \in V$, $\{f_j(l)\}_{j=1}^\infty$ converges to $f(l)$ as a sequence of real numbers in the standard topology on $\mathbb{R}$ for every $l \in \mathbb{Z}_+$, and there is an $n \geq 1$ such that $f_j(l) = f(l)$ when $j, l \geq n$, then it is easy to see that $\{f_j\}_{j=1}^\infty$ converges to $f$ in the topology defined on $V$. Conversely, suppose
that \( \{f_j\}_{j=1}^{\infty} \) is a sequence of elements of \( V \) which converges to \( f \in V \) in this topology. Let \( \rho \) be the nonnegative real-valued function on \( \mathbb{Z}_+ \) defined by

\[
\rho(n) = \frac{\sup\{\min(|f_j(l) - f(l)|, 1) : j, l \geq n\}}{2}.
\]

By construction, \( \rho \) is monotone decreasing on \( \mathbb{Z}_+ \). If \( \rho(n) > 0 \) for every \( n \in \mathbb{Z}_+ \), then there is an \( n_0 \geq 1 \) such that \( f_j \in N_\rho(f) \) when \( j \geq n_0 \), by convergence in \( V \). This implies that

\[
|f_j(l) - f(l)| < \rho(l) \leq \rho(n_0)
\]

when \( j, l \geq n_0 \), contradicting the definition of \( \rho(n_0) \). Therefore \( \rho(n) = 0 \) for some \( n \), which implies that \( f_j(l) = f(l) \) when \( j, l \geq n \). Also, \( \{f_j(l)\}_{j=1}^{\infty} \) converges to \( f(l) \) as a sequence of real numbers in the standard topology on \( \mathbb{R} \) for every \( l \in \mathbb{Z}_+ \), and so we are back in the situation described at the beginning of the paragraph. If \( V_0 \) is the space of \( f \in V \) such that \( f(l) \neq 0 \) for only finitely many \( l \), then \( V_0 \) is closed in \( V \). Indeed, if \( f(l) \neq 0 \) for infinitely many \( l \), then we can choose a positive function \( \rho \) on \( \mathbb{Z}_+ \) such that \( \rho(l) \leq |f(l)| \) when \( f(l) \neq 0 \), and every \( h \in N_\rho(f) \) satisfies \( h(l) \neq 0 \) when \( f(l) \neq 0 \), which is to say that \( N_\rho(f) \) is contained in the complement of \( V_0 \) in \( V \). The functions on \( \mathbb{Z}_+ \) with values in the rational numbers \( \mathbb{Q} \) are dense in \( V \), and similarly the \( \mathbb{Q} \)-valued functions in \( V_0 \) are dense in \( V_0 \).

### 14.1 Topologies and subspaces

Let \( X \) be a set, and suppose that \( X_1, X_2, \ldots \) is a sequence of subsets of \( X \) such that \( X_i \subseteq X_{i+1} \) for each \( i \) and \( X = \bigcup_{i=1}^{\infty} X_i \). Suppose too that each \( X_i \) is equipped with a topology \( \tau_i \), and that these topologies are compatible in the sense that the topology on \( X_i \) induced by \( \tau_{i+1} \) on \( X_{i+1} \) is the same as \( \tau_i \). It follows that the topology on \( X_i \) induced by \( \tau_i \) on \( X_i \) is the same as \( \tau_i \) when \( l > i \). Let us say that \( U \subseteq X \) is an open set in \( X \) if \( U \cap X_i \) is an open set in \( X_i \) with respect to \( \tau_i \) for every \( i \geq 1 \). It is easy to check that this defines a topology \( \tau \) on \( X \), using the fact that each \( \tau_i \) is a topology on \( X_i \). If \( U_i \subseteq X_i \) is an open set with respect to \( \tau_i \), then the compatibility condition implies that there is an open set \( U_{i+1} \subseteq X_{i+1} \) with respect to \( \tau_{i+1} \) such that \( U_i = U_{i+1} \cap X_i \). Repeating the process, for every \( l > i \) there is an open set \( U_l \subseteq X_l \) with respect to \( \tau_l \) such that \( U_l = U_l \cap X_l \). Hence \( U_l = U_n \cap X_l \) when \( n \geq l \geq i \). If \( U = \bigcup_{i=1}^{\infty} U_i \), then \( U_n = U \cap X_n \) for every \( n \geq i \). In particular, \( U \) is an open set in \( X \), and it follows that \( \tau_i \) is the same as the topology induced on \( X_i \) by \( \tau \).

As a variant of this construction, suppose also that \( X \) is a vector space over the real numbers, and that each \( X_i \) is a linear subspace of \( X \). Under suitable conditions, one might first say that a convex set \( U \subseteq X \) is an open set in \( X \) if \( U \cap X_i \) is an open set in \( X_i \) for every \( i \geq 1 \). One can then use the convex open sets to generate a topology on \( X \) in the usual way, i.e., where \( V \subseteq X \) is an open set if for every \( x \in V \) there is a convex open set \( U \subseteq X \) such that \( x \in U \) and \( U \subseteq V \).
15 Countable sets

Let $A$ be a set, and suppose that there is a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of $A$ in which every element of $A$ appears at least once. It may be that $A$ has only finitely many elements, and that some elements are repeated infinitely often in the sequence. At any rate, one can systematically go through the sequence and skip terms that are repeated. If $A$ has infinitely many elements, then this will leave a sequence $\{y_l\}_{l=1}^{\infty}$ of elements of $A$ in which every element of $A$ occurs exactly once. In this event we say that $A$ is countably infinite. If $B \subseteq A$ has infinitely many elements, then we can also go through the sequence and just keep the terms in $B$, to get a sequence $\{z_n\}_{n=1}^{\infty}$ of elements of $B$ in which every element of $B$ appears exactly once. It follows that $B$ is countably infinite too.

If one adds an element to a countably-infinite set, then the resulting set is countably-infinite, because one can get an enumeration of the elements in the new set by putting the new element in the first term and sliding the other terms over one step. Similarly, the union of a countably-infinite set with a set with only finitely many elements is countably infinite. One can enumerate the elements of a union of two countably-infinite sets by a sequence by using the terms indexed by even integers for one set and the terms indexed by odd integers for the other set. If $A_1, A_2, \ldots$, is a sequence of sets with only finitely many elements, then one can make a sequence listing the elements of the union $A = \bigcup_{j=1}^{\infty} A_j$ by first listing the elements of $A_1$, then the elements of $A_2$, etc., and hence $A$ has only finitely many elements or is countably infinite. The Cartesian product $\mathbb{Z}_+ \times \mathbb{Z}_+$, consisting of ordered pairs of positive integers, can be expressed as a union of a sequence of finite subsets, and is therefore countable infinite. Using this one can check that the union of a sequence of countably-infinite sets is countably-infinite. Basically, one can list the elements of the union by a doubly-indexed sequence, and convert that into a sequence of the usual type using an enumeration of $\mathbb{Z}_+ \times \mathbb{Z}_+$.

The set $\mathbb{Z}_+$ of positive integers can be enumerated trivially with the sequence $x_j = j$ and hence is countably infinite. The set $\mathbb{Z}$ of all integers can be enumerated by the sequence $y_l = l$, $y_{2l+1} = -l$ and is also countably infinite. For every positive integer $n$, the set of integer multiples of $1/n$ is basically the same as the set of all integers and therefore countably infinite, and consequently the set $\mathbb{Q}$ of rational numbers is countably infinite, being the union of these sets over all $n \in \mathbb{Z}_+$. Let $B$ be the set of functions $f$ on $\mathbb{Z}_+$ such that $f(l) = 0$ or 1 for every $l \in \mathbb{Z}_+$. Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of functions in $B$. If $f$ is the function on $\mathbb{Z}_+$ such that $f(l) = 1$ when $f_j(l) = 0$ and $f(l) = 0$ when $f_j(l) = 1$, then $f \in B$ and $f$ is different from $f_j$ for every positive integer $j$. It follows that $B$ is uncountable, which means that it is neither finite nor countably infinite. One can use this to show that the real line is uncountable. Specifically, every element of $B$ corresponds to a real number in the interval $[0, 1]$, using the element of $B$ as a binary expansion. Every element of $[0, 1]$ occurs in this way at least once and at most twice. If there were a sequence listing all of the real numbers in $[0, 1]$, then $B$ would be countably infinite as well, a contradiction.
15.1 The axiom of choice

Let $A$ be a set, and suppose that $f$ is a function on $A$ which assigns to every $a \in A$ a set $f(a) \subseteq A$. Let $B$ be the set of $a \in A$ such that $a \in B$ when $a \notin f(a)$ and $a \notin B$ when $a \in f(a)$. By construction, $B \neq f(a)$ for every $a \in A$, since $a$ is an element of exactly one of $B$, $f(a)$ for every $a \in A$. This generalizes the uncountability of the set of all binary sequences.

The axiom of choice can be formulated as the statement that for any set $A$ there is a mapping from the collection of all nonempty subsets of $A$ into $A$ which associates to every $B \subseteq A$ with $B \neq \emptyset$ an element $b$ of $A$ such that $b \in B$. As another common formulation, suppose that $I$ is a nonempty set and that for every $i \in I$, $A_i$ is a nonempty set. That is to say, $\{A_i\}_{i \in I}$ is a nonempty family of nonempty sets. The axiom of choice asserts that there is a function $f$ on $I$ with values in $\bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for every $i \in I$. The generalized Cartesian product $\prod_{i \in I} A_i$ of the $A_i$’s is customarily defined as the set of such functions $f$, and the axiom of choice is then the claim that this generalized Cartesian product is nonempty when $I$ and each $A_i$, $i \in I$, are nonempty.

15.2 Strong limit points

Let $X$ be a topological space, and suppose that $E \subseteq X$ and $p \in X$. Let us say that $p$ is a strong limit point of $E$ if for every open set $U \subseteq X$ with $p \in U$ there are infinitely many elements of $E$ in $U$. In particular, there is a $q \in E \cap U$ such that $q \neq p$ under these conditions, which implies that a strong limit point is automatically a limit point in the usual sense. Conversely, if $X$ satisfies the first separation condition, then one can check that an ordinary limit point is automatically a strong limit point. In any topological space $X$, the set of strong limit points of a set $E \subseteq X$ is a closed set.

Similarly, a point $p \in X$ is a condensation point of a set $E \subseteq X$ if for every open set $U \subseteq X$ with $p \in U$, $E \cap U$ is uncountable. Hence a condensation point of $E$ is a strong limit point. Again one can check that the set of condensation points of $E$ is a closed set in $X$.

16 Bases

Let $X$ be a topological space. A collection $\mathcal{B}$ of open subsets of $X$ is said to be a base for the topology of $X$ if every open set $V \subseteq X$ can be expressed as a union of open subsets of $X$ in $\mathcal{B}$. Equivalently, $\mathcal{B}$ is a base for the topology of $X$ if for every open set $V \subseteq X$ and every $x \in V$ there is a $U \in \mathcal{B}$ such that $x \in U$ and $U \subseteq V$. This is the same as saying that for every $p \in X$, the collection $\mathcal{B}(p)$ of $U \in \mathcal{B}$ such that $p \in U$ is a local base for the topology of $X$ at $p$. For example, the collection of open intervals $(a,b)$ with $a,b \in \mathbb{R}$, $a < b$, is a base for the standard topology of the real line. If $X$ is any set, then the collection of one-element subsets $\{x\}$, $x \in X$, is a base for the discrete topology on $X$. If $M$ is a metric space, then the collection of open balls in $M$ with arbitrary centers and radii form a base for the topology associated to the metric.
The existence of a base for the topology with only finitely or countably many elements is a very interesting condition on a topological space. For instance, the collection of open intervals \((a, b)\) with \(a, b \in \mathbb{Q}, a < b\), forms a countable base for the standard topology on the real line. More generally, suppose that \((M, d(x, y))\) is a metric space and that \(\{x_j\}_{j=1}^{\infty}\) is a sequence of points in \(M\) such that the set \(E\) consisting of all of the \(x_j\)'s, \(j \in \mathbb{Z}_+\), is dense in \(M\). In this event the family \(B(x_j, 1/l), j, l \in \mathbb{Z}_+\), of open balls forms a base for the topology of \(M\), which can be rearranged into an ordinary sequence using the countability of \(\mathbb{Z}_+ \times \mathbb{Z}_+\). Conversely, let \(X\) be any topological space with a base \(B\) for its topology. For every \(U \in B\) with \(U \neq \emptyset\), pick a point \(x(U) \in U\), and let \(E\) be the set of \(x(U), U \in B, U \neq \emptyset\). Because \(B\) is a base for the topology of \(X, E\) is dense in \(X\). If \(B\) has only finitely or countably many elements, then \(E\) has only finitely or countably many elements.

### 16.1 Sub-bases

Let \(X\) be a topological space, and let \(B_0\) be a collection of open subsets of \(X\). Consider the collection \(B_1\) of subsets of \(X\) of the form \(U_1 \cap \cdots \cap U_n\), where \(U_1, \ldots, U_n\) are finitely many elements of \(B_0\). We say that \(B_0\) is a sub-base for the topology of \(X\) if \(B_1\) is a base for the topology of \(X\). For example, the collection of open half-lines \((-\infty, a)\) and \((b, +\infty)\) for \(a, b \in \mathbb{R}\) is a sub-base for the standard topology on the real line, and it suffices to take \(a, b\) to be rational numbers. If \(B_0\) is any collection of subsets of a set \(X\) such that \(\bigcup \{U : U \in B_0\} = X\), then it is easy to see that \(B_0\) is a sub-base for a topology on \(X\), in which the open subsets of \(X\) are unions of elements of the corresponding collection \(B_1\).

Note that the collection of finite subsets of the set \(\mathbb{Z}_+\) of positive integer is countably infinite. Indeed, every finite subset of \(\mathbb{Z}_+\) is a subset of \(\{1, \ldots, n\}\) for some \(n \in \mathbb{Z}_+\), and of course there are only finitely many subsets of \(\{1, \ldots, n\}\) for each \(n \in \mathbb{Z}_+\). This shows that the collection of finite subsets of \(\mathbb{Z}_+\) can be expressed as the union of a sequence of finite sets, and hence is countable, and it follows that the collection of finite subsets of any countable set is countable too. If \(B_0\) is a collection of only finitely or countably many subsets of a set \(X\) and \(B_1\) is obtained from \(B_0\) as in the previous paragraph, then the preceding observation implies that \(B_1\) also has only finitely or countably many elements. If a topological space \(X\) has a sub-base \(B_0\) for its topology with only finitely or countably many elements, it therefore has a base \(B_1\) for its topology with only finitely or countably many elements as well.

### 16.2 Totally bounded sets

Let \((M, d(x, y))\) be a metric space. A set \(E \subseteq M\) is totally bounded if for every \(\epsilon > 0\) there is a finite set \(A_\epsilon \subseteq M\) such that

\[
E \subseteq \bigcup_{x \in A_\epsilon} B(x, \epsilon).
\]
Thus a totally bounded set is bounded, since the union of finitely many bounded subsets of $M$ is bounded. Suppose that $A_e \subseteq M$ satisfies (16.1). For every $x \in A_e$ with $B(x, \varepsilon) \cap E \neq \emptyset$, choose $y(x) \in E$ such that $d(y(x), x) < \varepsilon$. Let $\tilde{A}_\varepsilon$ be the set of $y(x)$’s obtained in this way. By construction, $\tilde{A}_\varepsilon \subseteq E$ and

\begin{equation}
E \subseteq \bigcup_{y \in \tilde{A}_\varepsilon} B(y, 2\varepsilon).
\end{equation}

Clearly the number of elements of $\tilde{A}_\varepsilon$ is less than or equal to the number of elements of $A_e$ when $A_e$ has only finitely many elements. Therefore, without loss of generality, we may suppose that $A_e \subseteq E$ in the definition of a totally bounded set. If $M$ is totally bounded, then $M$ is separable, in the sense that there is a dense set $A \subseteq M$ with only finitely or countably many elements, i.e., $A = \bigcup_{n=1}^{\infty} A_{1/n}$.

17 More examples

The exotic topologies $\tau_+, \tau_-$ on the real line can be characterized by the bases $B_+, B_-$ for their topologies, respectively, where $B_+$ consists of the intervals of the form $[a, b)$ and $B_-$ consists of the intervals of the form $(a, b]$, with $a, b \in \mathbb{R}$ and $a < b$. It has been mentioned previously that the rational numbers are dense in the real line with respect to each of these exotic topologies, and that each of these topologies has a countable local base for the topology at every point. If $B'_+ \subseteq B_+$ is any base for the topology of $\mathbb{R}$ equipped with the topology $\tau_+$, then for every $x \in \mathbb{R}$ there is an element $U(x)$ of $B'_+$ which contains $x$ and is contained in $[x, +\infty)$. If $x, y \in \mathbb{R}$ and $x < y$, then $x \in U(x) \setminus U(y)$, and hence $U(x) \neq U(y)$. It follows that any base for the topology of $\mathbb{R}$ equipped with $\tau_+$ has to be uncountable, using the fact that the real line is uncountable. Similarly, any base for the $\mathbb{R}$ equipped with $\tau_-$ has to be uncountable. Consequently, there are no metrics on $\mathbb{R}$ for which the associated topologies are $\tau_+$ or $\tau_-$, since a metric space that contains a countable dense set has a countable base for its topology.

Each interval $[a, b)$ in $\mathbb{R}$ is an open set with respect to the topology $\tau_+$, and it is also closed, because its complement $(-\infty, a) \cup [b, +\infty)$ in the real line is an open set in the topology $\tau_+$. If $x \in \mathbb{R}$, $E \subseteq \mathbb{R}$ is closed with respect to the topology $\tau_+$, and $x \notin E$, then there is a $t \in \mathbb{R}$ such that $[x, t) \subseteq \mathbb{R} \setminus E$. Hence $(\mathbb{R}, \tau_+)$ is a regular topological space, since $[x, t), (-\infty, x) \cup [t, +\infty)$ are disjoint open sets in the topology that contain $x, E$, respectively. For the same reasons, $(\mathbb{R}, \tau_-)$ is a regular topological space.

18 Stronger topologies

Suppose that $\tau_1, \tau_2$ are topologies on a set $X$ such that $\tau_2$ contains $\tau_1$. Thus the open subsets of $X$ with respect to $\tau_1$ are also open subsets of $X$ with respect
to \( \tau_2 \). In this event one may say that \( \tau_2 \) is a stronger topology on \( X \) than \( \tau_1 \). If \( p \in X \) is a limit point of \( E \subseteq X \) with respect to \( \tau_2 \), then \( p \) is also a limit point of \( E \) with respect to \( \tau_1 \). A sequence or net of elements of \( X \) that converges with respect to \( \tau_2 \) automatically converges with respect to \( \tau_1 \), and with the same limit. If \( X \) satisfies the first separation condition with respect to \( \tau_1 \), then \( X \) automatically satisfies the first separation condition with respect to \( \tau_2 \). The analogous statement works for the Hausdorff property too. The same type of argument does not work for regularity, since a stronger topology has more closed sets as well.

### 18.1 Completely Hausdorff spaces

A topological space \( X \) is said to be completely Hausdorff if for every pair of elements \( p, q \) of \( X \) with \( p \neq q \) there are open subsets \( U, V \) of \( X \) such that \( p \in U \), \( q \in V \), and \( U \cap V = \emptyset \). This obviously implies that \( X \) is Hausdorff, since \( U \subseteq \overline{U}, \overline{V} \subseteq V \). If \( \tau_1, \tau_2 \) are topologies on \( X \) such that \( X \) is completely Hausdorff with respect to \( \tau_1 \) and \( \tau_1 \subseteq \tau_2 \), then it is easy to see that \( X \) is also completely Hausdorff with respect to \( \tau_2 \). It is also easy to check that subspaces of completely Hausdorff spaces are completely Hausdorff with respect to the induced topology.

A topological space \( X \) is regular if and only if it satisfies the first separation condition, and for each point \( p \in X \) and open set \( U \subseteq X \) with \( p \in U \) there is an open set \( W \subseteq X \) such that \( p \in W \) and \( \overline{W} \subseteq U \). In this formulation of regularity, \( X \setminus U \) corresponds to the closed subset of \( X \) that does not contain \( p \) in the earlier formulation. Using this more local description of regularity, it is easy to see that regular topological spaces are completely Hausdorff. Thus a completely Hausdorff topological space is also said to satisfy the separation condition number two-and-a-half, between the second and third separation conditions.

### 19 Normality

A topological space \( X \) satisfies the fourth separation condition, or is normal, if it satisfies the first separation condition, and if for every pair \( E_1, E_2 \) of disjoint closed subsets of \( X \) there are disjoint open subsets \( U_1, U_2 \) of \( X \) such that \( E_1 \subseteq U_1 \) and \( E_2 \subseteq U_2 \). A normal topological space \( X \) is Hausdorff and regular, since the first separation condition implies that subsets of \( X \) with one element are closed.

If \((M, d(x, y))\) is a metric space, then \( M \) is normal with respect to the topology associated to the metric. For let a pair \( E_1, E_2 \) of disjoint closed subsets of \( M \) be given. For every \( x \in E_1 \), choose a positive real number \( r_1(x) \) such that \( B(x, r_1(x)) \subseteq M \setminus E_2 \), which we can do because \( x \in M \setminus E_2 \) and \( M \setminus E_2 \) is an open set. Similarly, for every \( y \in E_2 \), choose an \( r_2(y) > 0 \) such that \( B(y, r_2(y)) \subseteq M \setminus E_1 \). Let \( U_1 \) be the union of the open balls \( B(x, r_1(x)/2), x \in E_1 \), and let \( U_2 \) be the union of the open balls \( B(y, r_2(y)/2), y \in E_2 \). Thus \( U_1, U_2 \) are open subsets of \( M \) which contain \( E_1, E_2 \), respectively, by construction. Suppose for the sake of a contradiction that there is a point \( z \in U_1 \cap U_2 \). Hence
there are \( x \in E_1 \) and \( y \in E_2 \) such that \( d(x, z) < r_1(x)/2 \) and \( d(y, z) < r_2(y)/2 \). By the triangle inequality,

\[
(19.1) \quad d(x, y) < \frac{r_1(x) + r_2(y)}{2}.
\]

However, \( r_1(x) \) and \( r_2(y) \) are each less than or equal to \( d(x, y) \), because of the way that they were chosen and the fact that \( x \in E_1, y \in E_2 \). This contradicts (19.1), and it follows that \( U_1, U_2 \) are disjoint, as desired.

Now consider the real line equipped with the exotic topology \( \tau_+ \), in which \( U \subseteq \mathbb{R} \) is an open set if and only if for every \( x \in U \) there is a \( t \in \mathbb{R}, t > x \), such that \( [x, t] \subseteq U \). Let \( E_1, E_2 \) be disjoint closed subsets of \( \mathbb{R} \) with respect to the topology \( \tau_+ \). For every \( x \in E_1 \), there is a \( t_1(x) > x \) such that \( [x, t_1(x)] \subseteq \mathbb{R} \setminus E_2 \), since \( \mathbb{R} \setminus E_2 \) is an open set with respect to \( \tau_+ \) which contains \( x \). Similarly, for every \( y \in E_2 \) there is a \( t_2(y) > y \) such that \( [y, t_2(y)] \subseteq \mathbb{R} \setminus E_1 \). Let \( U_1 \) be the union of the intervals \( [x, t_1(x)] \), \( x \in E_1 \), and let \( U_2 \) be the union of the intervals \( [y, t_2(y)] \), \( y \in E_2 \). By construction, \( U_1, U_2 \) are open subsets of \( \mathbb{R} \) with respect to \( \tau_+ \) which contain \( E_1, E_2 \), respectively. Suppose for the sake of a contradiction that there is a point \( z \in U_1 \cap U_2 \). Hence there are \( x \in E_1 \) and \( y \in E_2 \) such that \( z \) is an element of both \( [x, t_1(x)] \) and \( [y, t_2(y)] \). Equivalently, \( x, y \leq z \) and \( z < t_1(x), t_2(y) \). If \( x \leq y \), then \( y \in [x, t_1(x)] \), contradicting the way that \( t_1(x) \) was chosen, since \( y \in E_2 \). For the same reasons, \( y \leq x \) is impossible, which implies that \( U_1 \cap U_2 = \emptyset \). This shows that \((\mathbb{R}, \tau_+) \) is a normal topological space, and \((\mathbb{R}, \tau_-) \) is a normal topological space as well, where \( \tau_- \) is the exotic topology generated by intervals of the form \((a, b]\).

19.1 Some remarks about subspaces

Let \( X \) be a topological space, let \( Y \) be a subset of \( X \), and let \( A_1, A_2 \) be disjoint relatively closed subsets of \( Y \). Because \( A_1 \), \( A_2 \) are relatively closed in \( Y \), there are closed subsets \( E_1, E_2 \) of \( X \) such that \( A_1 = E_1 \cap Y, A_2 = E_2 \cap Y \). However, one may not be able to take \( E_1, E_2 \) to be disjoint in \( X \), even though \( A_1, A_2 \) are disjoint in \( Y \). For example, if \( X \) is the real line with the standard topology, \( Y = \mathbb{R} \setminus \{0\} \), \( A_1 = (-\infty, 0) \), and \( A_2 = (0, +\infty) \), then \( A_1, A_2 \) are relatively closed subsets of \( \mathbb{R} \setminus \{0\} \), and \( E_1 = (-\infty, 0], E_2 = [0, +\infty) \) are the only closed subsets of \( \mathbb{R} \setminus \{0\} \) whose intersections with \( \mathbb{R} \setminus \{0\} \) are equal to \( A_1, A_2 \), respectively.

This indicates that the naive argument for showing that a subspace of a normal topological space is also normal with respect to the induced topology does not work. Of course, there is no problem in the example mentioned in the previous paragraph, since \( A_1, A_2 \) are already open sets as well as being relatively closed in \( \mathbb{R} \setminus \{0\} \). We also know that \( \mathbb{R} \setminus \{0\} \) is normal with respect to the induced topology, because the induced topology is the same as the topology induced by the restriction of the standard metric on \( \mathbb{R} \) to \( \mathbb{R} \setminus \{0\} \). Similarly, any subspace \( Y \) of a metric space \((X, d(x, y))\) is normal with respect to the induced topology, because the induced topology on \( Y \) is the same as the topology determined by the restriction of the metric \( d(x, y) \) to \( x, y \in Y \).
19.2 Another separation condition

A pair of subsets $A, B$ of a topological space $X$ are said to be separated if

$$(19.2) \quad A \cap B = \emptyset.$$ 

We say that $X$ satisfies the fifth separation condition, or is completely normal, if $X$ satisfies the first separation condition, and if for every pair $A, B$ of separated subsets of $X$ there are disjoint open subsets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. This implies that $X$ is normal, since disjoint closed sets are automatically separated. One can check that metric spaces satisfy the fifth separation condition, for basically the same reasons as before.

If $A, B \subseteq Y \subseteq X$, then it is easy to see that $A, B$ are separated as subsets of $X$ if and only if they are separated as subsets of $Y$, with respect to the topology on $Y$ induced by the given topology on $X$. This is because a point $p \in Y$ is adherent to $A$ as a subset of $X$ if and only if $p$ is adherent to $A$ as a subset of $Y$, with respect to the induced topology. If $X$ satisfies the fifth separation condition, then it follows that $Y$ does too, since one can apply the separation condition on $X$ directly to separated subsets of $Y$.

20 Continuous mappings

Suppose that $X, Y$ are topological spaces and that $f : X \to Y$ is a function defined on $X$ with values in $Y$, which is the same as a mapping from $X$ to $Y$. We say that $f$ is continuous at a point $p \in X$ if for every open set $V \subseteq Y$ with $f(p) \in V$ there is an open set $U \subseteq X$ such that $p \in U$ and $f(x) \in V$ for every $x \in U$. If $f$ is continuous at $p$, $(A, \prec)$ is a directed system, and $\{x_a\}_{a \in A}$ is a net that converges to $p$ in $X$, then the net $\{f(x)\}_{a \in A}$ converges to $f(p)$ in $Y$. Conversely, if $f$ is not continuous at $p$, then there is an open set $V \subseteq Y$ such that $f(p) \in V$ and for every open set $U \subseteq X$ with $p \in U$ there is a point $x(U) \in U$ such that $f(x(U)) \notin V$. The $x(U)$'s form a net indexed by the open subsets of $X$ that contain $p$ ordered by reverse inclusion, and this net converges to $p$ by construction. The $f(x(U))$'s form a net in $Y$ indexed in the same way that does not converge to $f(p)$. Therefore $f$ is continuous at $p$ if $f$ maps every convergent net in $X$ with limit $p$ to a convergent net in $Y$ with limit $f(p)$.

We say that $f$ is a continuous mapping from $X$ to $Y$ if $f$ is continuous at every point $p \in X$. If $f : X \to Y$ is continuous and $W \subseteq Y$ is an open set, then

$$f^{-1}(W) = \{x \in X : f(x) \in W\}$$

is an open set in $X$, because for every $x \in f^{-1}(W)$ there is an open set $U$ in $X$ such that $x \in U$ and $U \subseteq f^{-1}(W)$. It is easy to see from the definition of continuity that every mapping with this property is continuous. For any mapping $f : X \to Y$, one can check that

$$X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$$
for every \( A \subseteq Y \), and hence that \( f \) is continuous if and only if \( f^{-1}(A) \) is closed in \( X \) for every closed set \( A \subseteq Y \).

### 20.1 Simple examples

Here are some simple examples of continuous mappings \( f \) from a topological space \( X \) into a topological space \( Y \).

1. If \( X \) is equipped with the discrete topology, then any mapping \( f \) from \( X \) into any topological space \( Y \) is continuous.
2. If \( Y \) is equipped with the indiscrete topology, then any mapping \( f \) from any topological space \( X \) into \( Y \) is continuous.
3. If \( X = Y \) is the real line with the standard topology, then continuity is equivalent to the familiar definition in terms of \( \epsilon \)'s and \( \delta \)'s. Similarly, if \( X \) and \( Y \) are metric spaces, then continuity is equivalent to an analogous definition in terms of \( \epsilon \)'s and \( \delta \)'s.
4. Constant mappings between arbitrary topological spaces are continuous.
5. If \( X = Y \) with the same topology, then the identity mapping \( f(x) = x \) is continuous.
6. If \( X = Y \) as sets, and \( \tau_1, \tau_2 \) are topologies on \( X \) such that \( \tau_2 \subseteq \tau_1 \), then the identity mapping is continuous as a mapping from \( (X, \tau_1) \) to \( (X, \tau_2) \).
7. If \( X \subseteq Y \) and \( X \) is equipped with the topology induced from the one on \( Y \), then the standard embedding \( f(x) = x \) is continuous as a mapping from \( X \) into \( Y \).
8. If \( X \) is equipped with the indiscrete topology and \( Y \) satisfies the first separation condition, then every continuous mapping from \( X \) into \( Y \) is constant.
9. A mapping \( f : X \to Y \) is said to be **locally constant** if for every \( p \in X \) there is an open set \( U \subseteq X \) such that \( p \in U \) and \( f \) is constant on \( U \). Locally constant mappings are always continuous.
10. If \( Y \) is equipped with the discrete topology, then every continuous mapping from \( X \) into \( Y \) is locally constant.

### 20.2 Sequentially continuous mappings

Let \( X, Y \) be topological spaces, and let \( f \) be a mapping from \( X \) to \( Y \). We say that \( f \) is **sequentially continuous** at a point \( p \in X \) if for every sequence \( \{x_i\}_{i=1}^{\infty} \) of elements of \( X \) which converges to \( p \) in \( X \), \( \{f(x_i)\}_{i=1}^{\infty} \) converges to \( f(p) \) as a sequence in \( Y \). If \( f \) is continuous at \( p \) in the usual sense, then \( f \) is sequentially continuous at \( p \), because sequences are special cases of nets. Conversely, sequential continuity at \( p \) implies continuity at \( p \) when the domain satisfies the local countability condition at \( p \). Similarly, \( f : X \to Y \) is **sequentially continuous** if it is sequentially continuous at every point \( p \in X \). Hence continuous mappings are sequentially continuous. If \( X \) satisfies the local countability condition at each point, then every sequentially continuous mapping \( f : X \to Y \) is continuous.
21 The product topology

Suppose that \((X_1, \tau_1), (X_2, \tau_2)\) are topological spaces. Consider the Cartesian product \(X_1 \times X_2\), consisting of all ordered pairs \((x_1, x_2)\) with \(x_1 \in X_1\) and \(x_2 \in X_2\). The product topology \(\tau_1 \times \tau_2\) on \(X_1 \times X_2\) can be defined by saying that \(W \subseteq X_1 \times X_2\) is an open set if and only if for every \((x_1, x_2)\in W\) there are open sets \(U_1 \subseteq X_1, U_2 \subseteq X_2\) such that \(x_1 \in U_1, x_2 \in U_2\), and \(U_1 \times U_2 \subseteq W\).

Equivalently, If \(U \subseteq X_1\) and \(V \subseteq X_2\) are open sets, then \(U \times V\) is an open set in \(X_1 \times X_2\), and products of open sets like these form a base for the product topology on \(X_1 \times X_2\). If \((A, \prec)\) is a directed system and \(\{(x_{1,a}, x_{2,a})\}_{a \in A}\) is a net in \(X_1 \times X_2\), then this net converges to \((x_1, x_2)\in X_1 \times X_2\) in the product topology if and only if the nets \(\{x_{1,a}\}_{a \in A}, \{x_{2,a}\}_{a \in A}\) converge to \(x_1, x_2\), respectively, in \(X_1, X_2\).

Let \(p_1 : X_1 \times X_2 \to X_1, p_2 : X_1 \times X_2\) be the two coordinate projections associated to the Cartesian product, which is to say that

\[p_1(x_1, x_2) = x_1, \quad p_2(x_1, x_2) = x_2\]

for every \((x_1, x_2) \in X_1 \times X_2\). It is easy to see from the definition of the product topology that \(p_1, p_2\) are automatically continuous as mappings from \(X_1 \times X_2\) with the product topology to \(X_1, X_2\) with their given topologies, respectively. In general, if \(X, Y\) are topological spaces and \(f\) is a mapping from \(X\) to \(Y\), then \(f : X \to Y\) is an open mapping if

\[f(U) = \{y \in Y : y = f(x) \text{ for some } x \in U\}\]

is an open set in \(Y\) for every open set \(U \subseteq X\). One can check that the coordinate mappings \(p_1, p_2\) are automatically open mappings too.

If \(E_1 \subseteq X_1\) and \(E_2 \subseteq X_2\) are closed sets, then \(E_1 \times E_2\) is a closed set in \(X_1 \times X_2\) with respect to the product topology. This is because

\[(X_1 \times X_2) \setminus (E_1 \times E_2) = ((X_1 \setminus E_1) \times X_2) \cup (X_1 \times (X_2 \setminus E_2)),\]

which is the union of two open subsets of \(X_1 \times X_2\) with respect to the product topology. If \(X_1\) and \(X_2\) both satisfy the first, second, or third separation condition, then one can check that \(X_1 \times X_2\) satisfies the same condition with respect to the product topology. If \(B_1, B_2\) are bases for the topologies of \(X_1, X_2\), respectively, then

\[B = \{U_1 \times U_2 : U_1 \in B_1, U_2 \in B_2\}\]

is a base for the product topology on \(X_1 \times X_2\). In particular, if \(B_1, B_2\) have only finitely or countably many elements, then there are only finitely or countably many elements in \(B\) as well. There are analogous statements for local bases for the topologies of \(X_1, X_2\) at individual elements, and for the local countability condition. If the topologies on \(X_1, X_2\) are determined by metrics \(d_1(x_1, y_1), d_2(x_2, y_2)\), then it is easy to see that

\[d(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2)), \quad x = (x_1, y_1), y = (y_1, y_2) \in X_1 \times X_2,\]

is a metric on \(X_1 \times X_2\) that determines the product topology on \(X_1 \times X_2\).
21.1 Countable products

If \( X_1, X_2, \ldots \), is a sequence of sets, then their Cartesian product \( X = \prod_{i=1}^{\infty} X_i \) can be defined as the set of sequences \( x = \{x_i\}_{i=1}^{\infty} \) such that \( x_i \in X_i \) for every \( i \geq 1 \). Suppose that each \( X_i \) is equipped with a topology \( \tau_i \) too. The corresponding product topology \( \tau \) on \( X \) is defined by saying that \( U \subseteq X \) is an open set if for every \( x \in U \) there are open sets \( U_i \subseteq X_i \), for every \( i \geq 1 \), \( \prod_{i=1}^{\infty} U_i \subseteq U \), and \( U_i = X_i \) for all but finitely many \( i \). Another topology \( \tau' \) on \( X \) can be defined in the same way but without the restriction that \( U_i = X_i \) for all but finitely many \( i \), which means that the open subsets of \( X \) with respect to \( \tau' \) are much smaller in general than for the product topology. We shall call this the strong product topology on \( X \), to distinguish it from the ordinary product topology.

With respect to both the ordinary and strong product topologies on \( X \), the coordinate projections \( p_i : X \to X_i \) defined by \( p_i(x) = x_i \) are continuous open mappings for each \( i \). The product topology on \( X \) may be described as the weakest topology on \( X \) for which these coordinate mappings are continuous for each \( i \). If \( E_i \subseteq X_i \) is a closed set for each \( i \), then \( E = \prod_{i=1}^{\infty} E_i \) is a closed set in \( X \) with respect to the product topology, and hence also with respect to the strong product topology, because \( X \setminus E \) can be expressed as the union of open sets, and is therefore open. If \( X_i \) satisfies the first, second, or third separation condition for each \( i \), then one can check that \( X \) also satisfies the same condition, with respect to both the ordinary and strong product topologies. If \( X_i \) is equipped with the discrete topology for each \( i \), then the strong product topology on \( X \) is the discrete topology too.

Let \( \mathcal{B}_i \) be a base for the topology of \( X_i \) for each \( i \), and let \( \mathcal{B}(n) \) be the collection of open subsets of \( X \) with respect to the product topology of the form \( \prod_{i=1}^{n} U_i \), where \( U_i \in \mathcal{B}_i \) when \( i \leq n \) and \( U_i = X_i \) when \( i > n \). It is easy to see that

\[
\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}(n)
\]

is a base for the product topology on \( X \). If \( \mathcal{B}_i \) has only finitely or countably many elements for each \( i \), then \( \mathcal{B}(n) \) has only finitely or countably many elements for each \( n \), and hence \( \mathcal{B} \) has only finitely or countably many elements as well. As before, there are analogous statements for local bases and the local countability condition. Suppose that each \( X_i \) is equipped with a metric \( d_i(\cdot, \cdot) \) that determines the topology \( \tau_i \). Without loss of generality, we may suppose also that \( d_i(p, q) \leq 1/i \) for every \( p, q \in X_i \) and \( i \geq 1 \), since otherwise we can replace \( d_i(p, q) \) with \( \min(d_i(p, q), 1/i) \). In this case, one can check that

\[
d(x, y) = \sup \{d_i(x_i, y_i) : i \geq 1\}
\]

is a metric on \( X \) that determines the product topology on \( X \).

30
21.2 Arbitrary products

Let $I$ be a set, and suppose that for every $i \in I$ we have a set $X_i$. Let $X$ be the generalized Cartesian product $\prod_{i \in I} X_i$ of the $X_i$’s, which is to say the set of functions $f$ defined in $I$ with values in $\bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for every $i \in I$. Suppose that each $X_i$ is equipped with a topology $\tau_i$. The product topology on $X$ is defined by saying that $U \subseteq X$ is an open set if for every $f \in U$ there are open sets $U_i \subseteq X_i$, $i \in I$, such that $f(i) \in U_i$ for every $i \in I$, $\prod_{i \in I} U_i \subseteq U$, and $U_i = X$ for all but finitely many $i \in I$. The condition that $f(i) \in U_i$ for every $i \in I$ is equivalent to $f \in \prod_{i \in I} U_i$. One can get another topology with much smaller open subsets in general by allowing the $U_i$’s to be arbitrary open subsets of the $X_i$’s, without the requirement that $U_i = X_i$ for all but finitely many $i$. As in the previous section, we shall call this the strong product topology on $X$.

The coordinate projections $p_i : X \to X_i$ are defined by $p_i(f) = f(i)$ for each $i \in I$, and are continuous open mappings with respect to the ordinary and strong product topologies on $X$. The product topology on $X$ is again the weakest topology for which these coordinate mappings are all continuous. If $E_i \subseteq X_i$ is a closed set for each $i \in I$, then $E = \prod_{i \in I} E_i$ is a closed set in $X$ with respect to the product topology, and hence with respect to the strong product topology, because $X \setminus E$ can be expressed as the union of open set, and is therefore open. As before, one can check that $X$ satisfies the first, second, or third separation condition when $X_i$ satisfies the same condition for each $i \in I$. If $X_i$ is equipped with the discrete topology for each $i \in I$, then the strong product topology on $X$ is again the discrete topology.

22 Subsets of metric spaces

Let $(M, d(x, y))$ be a metric space. If $A \subseteq M$ and $r > 0$, then put

$$A_r = \bigcup_{x \in A} B(x, r),$$

which is the same as the set of $y \in M$ for which there is an $x \in A$ such that $d(x, y) < r$. By construction,

$$A \subseteq A_r,$$

and $A_r$ is an open set in $M$ for every $r > 0$, being a union of open sets. Observe that

$$A_r \subseteq A_t$$

when $r < t$. Furthermore,

$$\bigcap_{r > 0} A_r = \overline{A}.$$

Specifically, $y \in M$ is an element of the closure $\overline{A}$ of $A$ if and only if $y$ is adherent to $A$, which means that for every $r > 0$ there is an $x \in A$ such that $d(x, y) < r$, and which is the same as saying that $y \in A_r$ for every $r > 0$. We can just as well
restrict our attention to \( r > 0 \) of the form \( 1/n \), where \( n \) is a positive integer, to get that
\[
\bigcap_{n=1}^{\infty} A_{1/n} = \overline{A}.
\]

In particular, every closed set in \( M \) can be expressed as the intersection of a sequence of open sets. Equivalently, every open set in \( M \) can be expressed as the union of a sequence of closed sets. If \( X \) is a topological space which satisfies the first separation condition, \( p \in X \), and \( X \) also satisfies the local countability condition at \( p \), then \( \{p\} \) can be expressed as the intersection of a sequence of open subsets of \( X \).

### 22.1 The Baire category theorem

Let \((M,d(x,y))\) be a complete metric space, and let \( U_1, U_2, \ldots \) be a sequence of dense open subsets of \( M \). The Baire category theorem states that the intersection \( \bigcap_{i=1}^{\infty} U_i \) is dense in \( M \) under these conditions. To see this, let \( x_0 \in M \) and \( r_0 > 0 \) be given, and consider the closed ball \( B_0 = \overline{B}(x_0, r_0) \). Because \( U_1 \) is a dense open set in \( M \), there is an \( x_1 \in U_1 \) and an \( r_1 > 0 \) such that \( B_1 = \overline{B}(x_1, r_1) \subseteq B_0 \cap U_1 \) and \( r_1 \leq 1 \). Similarly, there is an \( x_2 \in U_2 \) and an \( r_2 > 0 \) such that \( B_2 = \overline{B}(x_2, r_2) \subseteq B_1 \cap U_2 \) and \( r_2 \leq 1/2 \). In general, for every positive integer \( i \) there is an \( x_i \in U_i \) and an \( r_i > 0 \) such that \( B_i = \overline{B}(x_i, r_i) \subseteq B_{i-1} \cap U_1 \) and \( r_i \leq 1/i \). By construction, \( \{x_i\}_{i=1}^{\infty} \) is a Cauchy sequence in \( M \) and therefore converges to some point \( x \in M \). Since \( x_i \in B_l \) for every \( i \geq l \) and \( B_l \) is a closed ball, we get that \( x \in B_l \) for every \( l \geq 0 \). Hence \( x \in B_0 \) and \( x \in U_l \) for every \( l \geq 1 \), as desired. It follows from the Baire category theorem that the set of rational numbers is not the intersection of a sequence of open subsets of the real line.

### 22.2 Sequences of open sets

Let \( B \) be the set of all binary sequences, and remember that there is a one-to-one correspondence between \( B \) and the real line. There is also a one-to-one correspondence between \( B \) and the set of sequences of integers, or even the set of sequences of elements of \( B \).

If \( X \) is a topological space with a countable base for its topology, then the collection of open subsets of \( X \) has cardinal number less than or equal to that of \( B \). If \( X \) is any topological space for which the collection of open sets has cardinality less than or equal to that of \( B \), then the collection of sequences of open subsets of \( X \) also has cardinality less than or equal to the cardinality of \( B \). Hence the collection of subsets of \( X \) which can be expressed as the intersection of a sequence of open subsets of \( X \) has cardinality less than or equal to that of \( B \) in this case. However, the collection of all subsets of \( X \) may have larger cardinality. This happens when \( X \) is the real line, for instance, since the cardinality of the collection of all subsets of a set is always strictly larger than the cardinality of the set itself.
23 Open sets in $\mathbb{R}$

Let $U$ be a nonempty open set in the real line, with respect to the standard topology. For every $x \in U$, there are $a, b \in \mathbb{R}$ such that $a < x < b$ and $(a, b) \subseteq U$, since $U$ is an open set. Moreover, there are extended real numbers $a', b'$ with $a' < x < b'$ which satisfy the following three conditions:

\begin{align*}
(a', b') &\subseteq U; \\
a' &= -\infty \text{ or } a' \not\in U; \\
b' &= +\infty \text{ or } b' \not\in U.
\end{align*}

Specifically, $a'$ is the infimum of the $a < x$ such that $(a, x) \subseteq U$, and $b'$ is the supremum of the $b > x$ such that $(x, b) \subseteq U$. Any two open intervals satisfying (23.1), (23.2), and (23.3) are necessarily the same interval or are disjoint subsets of the real line. For any such open interval, one can pick a rational number in the interval, and different intervals are then associated to different rational numbers because of the disjointness of the intervals. It follows that every nonempty open set in the real line can be expressed as the union of a finite or countably-infinite collection of disjoint open intervals, where the intervals may be bounded or unbounded, depending on the situation.

One can consider similar matters for the real line equipped with the exotic topologies $\tau_+$, $\tau_-$. If $U \subseteq \mathbb{R}$ is an open set with respect to $\tau_+$ and $x \in U$, then there is a $b > x$ such that $[x, b) \subseteq U$. By taking the supremum of these $b$'s, it follows that there is an extended real number $b' > x$ such that $[x, b') \subseteq U$ and either $b' = +\infty$ or $b' \not\in U$. By taking the infimum of the $a \leq x$ such that $[a, x) \subseteq U$, one gets an extended real number $a' \leq x$ such that either $a' = -\infty$ and $(-\infty, x) \subseteq U$, or $-\infty < a' < x$, $(a', x) \subseteq U$, and $a' \not\in U$, or $-\infty < a' \leq x$, $[a', x) \subseteq U$, and $(c, x] \not\subseteq U$ when $c < a'$. There are analogous statements for the exotic topology $\tau_-$ based on intervals of the form $(a, b]$.

23.1 Collections of open sets

If $X$ is a topological space with a countable base for its topology, then the cardinality of the collection of open subsets of $X$ is less than or equal to the cardinality of the real line, because every open set in $X$ can be described by a sequence of elements of a countable base for the topology of $X$. In particular, the cardinality of the collection of open subsets of the real line is less than or equal to the cardinality of $\mathbb{R}$. Of course the cardinality of $\mathbb{R}$ is less than or equal to the cardinality of the set of all open subsets of $\mathbb{R}$, since $\mathbb{R} \setminus \{x\}$ is an open set in $\mathbb{R}$ for every $x \in \mathbb{R}$. One can also show that the cardinality of the collection of all open subsets of $\mathbb{R}$ is less than or equal to the cardinality of $\mathbb{R}$ using the fact that every open set in $\mathbb{R}$ is the union of a sequence of open intervals. Remember that the cardinality of the real line is the same as that of the set $\mathcal{B}$ of binary sequences. Also, the cardinality of $\mathcal{B} \times \mathcal{B}$ is equal to the cardinality of $\mathcal{B}$. It follows that the cardinality of the collection of all open intervals in $\mathbb{R}$ is less than or equal to the cardinality of $\mathcal{B}$. Hence the cardinality
of the collection of all open subsets of $\mathbb{R}$ is less than or equal to the cardinality of the set of sequences of elements of $\mathcal{B}$, and therefore less than or equal to the cardinality of $\mathcal{B}$. A similar argument can be applied to the real line equipped with the exotic topologies $\tau_+$ or $\tau_-$.  

24 Compactness

Let $X$ be a topological space and let $E \subseteq X$ be given. By an open covering of $E$ we mean a family $\{U_i\}_{i \in I}$ of open subsets of $X$ such that

$$E \subseteq \bigcup_{i \in I} U_i.$$  

We say that $E$ is compact in $X$ if for every open covering $\{U_i\}_{i \in I}$ of $E$ in $X$ there is a finite subcovering. This means that there are $i_1, \ldots, i_n \in I$ such that

$$E \subseteq U_{i_1} \cup \cdots \cup U_{i_n}.$$  

If $E$ has only finitely many elements, then $E$ is automatically compact. A famous theorem states that every closed interval $[a, b]$ in the real line is compact, with respect to the standard topology on $\mathbb{R}$. We shall not give the usual proof here, but instead we shall see later how this can be obtained as a consequence of general results about compactness.

If $Y \subseteq X$ and $E \subseteq Y$, then one can also consider compactness of $E$ relative to $Y$, using the topology induced from the one on $X$. A nice feature of compactness is that $E$ is compact relative to $X$ if and only if it is compact relative to $Y$, basically because every open covering of $E$ relative to $X$ can be converted to an open covering relative to $Y$ and vice-versa. Suppose that $\tau_1, \tau_2$ are topologies on $X$ such that the open subsets of $X$ with respect to $\tau_1$ are also open with respect to $\tau_2$. If $E \subseteq X$ is compact with respect to $\tau_1$, then $E$ is compact with respect to $\tau_2$, because every open covering of $E$ with respect to $\tau_1$ is an open covering with respect to $\tau_2$. If $X$ is equipped with the discrete topology and $E \subseteq X$ is compact, then $E$ has only finitely many elements.

24.1 A class of examples

Let $X$ be a topological space, and let $\{x_l\}_{l=1}^\infty$ be a sequence of elements of $X$ that converges to $x \in X$. Let $K$ be the set consisting of the $x_l$’s, $l \geq 1$, and $x$. It is easy to see that $K$ is a compact set in $X$. For suppose that $\{U_i\}_{i \in I}$ is an arbitrary covering of $K$ by open subsets of $X$. Since $x \in K$, there is an $i_0 \in I$ such that $x \in U_{i_0}$. Because $\{x_l\}_{l=1}^\infty$ converges to $x$ in $X$, there is an $L \geq 1$ such that $x_l \in U_{i_0}$ when $l \geq L$. If we choose $i_1, \ldots, i_{L-1} \in I$ such that $x_l \in U_{i_l}$ for $l = 1, \ldots, L-1$, then $K \subseteq U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_{L-1}}$, as desired. Suppose now that $X$ satisfies the local countability condition at every point. In this case, a set $E \subseteq X$ is closed in $X$ if and only if for every sequence $\{x_l\}_{l=1}^\infty$ of elements of $E$ which converges to a point $x \in X$ we have that $x \in E$. It follows from
the previous remarks that \( E \subseteq X \) is a closed set in \( X \) if and only if \( E \cap K \) is relatively closed in \( K \) for every compact set \( K \subseteq X \). Consequently, \( U \subseteq X \) is an open set in \( X \) if and only if \( U \cap K \) is relatively open in \( K \) for every compact set \( K \subseteq X \).

### 25 Properties of compact sets

Let \( X \) be a topological space, let \( K_1, K_2 \) be compact subsets of \( X \), and let us show that their union \( K_1 \cup K_2 \) is a compact set in \( X \). Let \( \{U_i\}_{i \in I} \) be any open covering of \( K_1 \cup K_2 \) in \( X \). Hence \( \{U_i\}_{i \in I} \) is an open covering of \( K_1 \) and \( K_2 \) individually, and it follows that there are finite subsets \( I_1, I_2 \) of \( I \) such that

\[
K_1 \subseteq \bigcup_{i \in I_1} U_i, \quad K_2 \subseteq \bigcup_{i \in I_2} U_i.
\]

Therefore \( I_1 \cup I_2 \subseteq I \) has only finitely many elements and

\[
K_1 \cup K_2 \subseteq \bigcup_{i \in I_1 \cup I_2} U_i.
\]

Thus there is a finite subcovering of \( K_1 \cup K_2 \) from the open covering \( \{U_i\}_{i \in I} \), and it follows that \( K_1 \cup K_2 \) is compact.

If \( K \subseteq X \) is compact and \( E \subseteq X \) is closed, then \( E \cap K \) is compact. For if \( \{U_i\}_{i \in I} \) is any open covering of \( E \cap K \) in \( X \), then the \( U_i \)'s together with \( X \setminus E \) form an open covering of \( K \), which implies that \( K \) is contained in the union of finitely many \( U_i \)'s and \( X \setminus E \). Clearly \( E \cap K \) can be covered by the same collection of finitely many \( U_i \)'s, and therefore \( E \cap K \) is compact. If \( X \) is Hausdorff and \( K \subseteq X \) is compact, then \( K \) is a closed set in \( X \). To see this, let \( p \in X \setminus K \) be given, and for every \( q \in K \) let \( U(p, q), V(p, q) \) be disjoint open subsets of \( X \) such that \( p \in U(p, q) \) and \( q \in V(p, q) \). Since \( K \) is compact, there are finitely many elements \( q_1, \ldots, q_n \) of \( K \) such that \( K \) is contained in the union \( V(p) \) of \( V(p, q_1), \ldots, V(p, q_n) \). If \( U(p) \) is the intersection of \( U(p, q_1), \ldots, U(p, q_n) \), then \( U(p) \) is an open set in \( X \) such that \( p \in U(p) \) and \( U(p) \cap V(p) = \emptyset \). In particular, \( U(p) \subseteq X \setminus K \), so that \( X \setminus K = \bigcup_{p \in K} U(p) \) is an open set in \( X \), and hence \( K \) is closed. By contrast, if \( X \) is equipped with the indiscrete topology, then every subset of \( X \) is compact. Similarly, if \( X \) is an infinite set equipped with the topology in which \( W \subseteq X \) is an open set exactly when \( W \) is the empty set or \( X \setminus W \) has only finitely many elements, then \( X \) satisfies the first separation condition, and one can check that every subset of \( X \) is again compact.

#### 25.1 Disjoint compact sets

Let \( X \) be a Hausdorff topological space. If \( p \in X \), \( K \subseteq X \) is compact, and \( p \in X \setminus K \), then there are open subsets \( U(p), V(p) \) of \( X \) such that \( p \in U(p) \), \( K \subseteq V(p) \), and \( U(p) \cap V(p) = \emptyset \), as before. Thus \( X \) satisfies the analogue of the third separation condition for compact sets instead of closed sets. If
$H \subseteq X$ is another compact set and $H \cap K = \emptyset$, then it follows that there are finitely many elements $p_1, \ldots, p_n$ of $H$ such that $H$ is contained in the union $U$ of $U(p_1), \ldots, U(p_n)$. If $V$ is the intersection of $V(p_1), \ldots, V(p_n)$, then $U, V$ are open subsets of $X$ such that $H \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$. Hence $X$ satisfies the analogue of the fourth separation condition for compact sets instead of closed sets. In particular, a compact Hausdorff topological space $X$ is normal, because closed subsets of $X$ are compact. Similarly, if $X$ is a regular topological space, $H \subseteq X$ is compact, $E \subseteq X$ is closed, and $H \cap E = \emptyset$, then $H, E$ are contained in disjoint open subsets of $X$. If $X$ is completely Hausdorff, then disjoint compact subsets of $X$ are contained in open sets with disjoint closures, by analogous arguments.

25.2 The limit point property

If $X$ is a topological space, then $E \subseteq X$ has the limit point property if every infinite set $A \subseteq E$ has a limit point in $E$. Similarly, $E \subseteq X$ has the strong limit point property if every infinite set $A \subseteq E$ has a strong limit point in $E$. The strong limit point property implies the limit point property, and the converse holds when $X$ satisfies the first separation condition, since every limit point is then a strong limit point. These two conditions are quite similar to compactness. For instance, if $E$ has only finitely many elements, then $E$ automatically has the strong limit point property. If $E \subseteq Y \subseteq X$, then $E$ satisfies one of these conditions relative to $Y$ if and only if it satisfies the same condition relative to $X$. If $E_1, E_2 \subseteq X$ satisfy the limit point property or the strong limit point property, then $E_1 \cup E_2$ has the same feature. For if $A \subseteq E_1 \cup E_2$ has infinitely many elements, then at least one of $A \cap E_1, A \cap E_2$ has infinitely many elements, and hence a limit point or strong limit point in $E_1$ or $E_2$. The intersection of a set which satisfies either of these versions of the limit point property with a closed set in $X$ also satisfies the same version of the limit point property, since a closed set contains all of its limit points and therefore all limit points of its subsets. If $K \subseteq X$ is compact, then $K$ satisfies the strong limit point property. For if $A \subseteq K$ is an infinite set without a strong limit point in $K$, then for every $p \in K$ there is an open set $U(p) \subseteq X$ such that $p \in U(p)$ and $A$ has only finitely many elements in $U(p)$. Compactness implies that there are finitely many elements $p_1, \ldots, p_n$ of $K$ such that $K \subseteq U(p_1) \cup \cdots \cup U(p_n)$. Hence $A$ has only finitely many elements, a contradiction. As a partial converse to this statement, suppose that $E \subseteq X$ has the strong limit point property and that $V_1, V_2, \ldots$ is a sequence of open subsets of $X$ such that $E \subseteq \bigcup_{i=1}^{\infty} V_i$. If for every $n \geq 1$ there is an $x_n \in E \setminus \bigcup_{i=1}^{n} V_i$, then the set $A \subseteq E$ of $x_n$’s has infinitely many elements. A strong limit point $p \in E$ of $A$ would be in $V_l$ for some $l$, which would imply that $x_n \in V_l$ for some $n > l$, a contradiction. It follows that $E \subseteq \bigcup_{i=1}^{n} V_i$ for some $n$. 

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Let $X$ be a topological space, let $\mathcal{B}$ be a base for the topology of $X$, and let $\{U_i\}_{i \in I}$ be any family of open subsets of $X$. Put
\[ B_i = \{V \in \mathcal{B} : V \subseteq U_i\} \]
for each $i \in I$, so that
\[ \bigcup_{V \in B_i} V = U_i. \]
If we put
\[ B' = \bigcup_{i \in I} B_i, \]
then we get that
\[ \bigcup_{V \in B'} V = \bigcup_{i \in I} \left( \bigcup_{V \in B_i} V \right) = \bigcup_{i \in I} U_i. \]
If $V \in B'$, then $V \in B_i$ for some $i \in I$, and we let $i(V)$ be an element of $I$ with this property. Thus $V \subseteq U_{i(V)}$ for each $V \in B'$, and we let $I_1$ be the set of $i(V)$ with $V \in B'$ chosen in this way. It follows that
\[ \bigcup_{V \in B'} V \subseteq \bigcup_{i \in I_1} U_{i(V)} = \bigcup_{i \in I_1} U_i, \]
so that
\[ \bigcup_{i \in I} U_i = \bigcup_{V \in B'} V \subseteq \bigcup_{i \in I_1} U_i. \]
This implies that
\[ \bigcup_{i \in I} U_i = \bigcup_{i \in I_1} U_i, \]
because the left side is automatically contained in the right side, since $I_1 \subseteq I$.
If $\mathcal{B}$ has only finitely or countably many elements, then $B' \subseteq \mathcal{B}$ has only finitely or countably many elements, and it is easy to see that $I_1$ has only finitely or countably many elements as well.

26.1 The Lindelöf property
A subset $E$ of a topological space $X$ is said to have the Lindelöf property if every open covering of $E$ in $X$ can be reduced to a finite or countable subcovering. Thus compact sets have the Lindelöf property, by definition. The Lindelöf property is often defined initially for topological spaces, and subsets of topological spaces are treated as topological spaces themselves, using the induced topology. This is equivalent to defining the Lindelöf property directly for subsets, as before, by an argument like the one for compact sets. If a subset $E$ of a topological space $X$ has both the Lindelöf and strong limit point properties, then it is easy
to see that $E$ is compact, because the strong limit point property implies that every countable open covering of $E$ has a finite subcovering.

Let $X$ be a topological space, and let $B$ be a base for the topology of $X$. If $Y$ is any subset of $X$, then the collection $B_Y$ of the sets of the form $U \cap Y$ with $U \in B$ is a base for the topology on $Y$ induced from the topology on $X$. In particular, if $B$ has only finitely or countably many elements, then $B_Y$ has only finitely or countably many elements as well. If $B$ has only finitely or countably many elements, then every subset of $X$ has the Lindelöf property, by the earlier argument. In this case, it follows that every subset of $X$ with the strong limit point property is compact.

### 26.2 Applications to metric spaces

Let $(M, d(x, y))$ be a metric space. If $E \subseteq M$ is compact, then it is easy to see that $E$ is totally bounded, by considering the covering of $E$ by open balls of radius $\epsilon$ centered at elements of $E$. Similarly, if $E$ has the Lindelöf property, then for every $\epsilon > 0$, $E$ can be covered by finitely or countably many open balls of radius $\epsilon$ centered at elements of $E$. In particular, if $M$ has the Lindelöf property, then $M$ is separable, since one can apply the previous statement to $\epsilon = 1/n$ for each $n \in \mathbb{Z}_+$. Conversely, if $M$ is separable, then there is a base for the topology of $M$ with only finitely or countably many elements, which implies that $M$ has the Lindelöf property, as before.

If $E \subseteq M$ has the limit point property, then $E$ is totally bounded. Otherwise, there is an $\epsilon > 0$ such that $E$ cannot be covered by finitely many balls with radius $\epsilon$. This implies that there is an infinite sequence $x_1, x_2, x_3, \ldots$ of elements of $E$ such that $d(x_j, x_l) \geq \epsilon$ when $j < l$. The set $A \subseteq E$ of these $x_j$'s has infinitely many elements and no limit points, which gives a contradiction. If $M$ has the limit point property, then $M$ is totally bounded, which implies that $M$ is separable, and hence that $M$ has the Lindelöf property. It follows that $M$ is compact, using the hypothesis that $M$ have the limit point property again. If $E \subseteq M$ has the limit point property, then $E$ is compact, since one can apply the previous argument to $E$, considered as a metric space itself, using the restriction of the metric $d(x, y)$ to $x, y \in E$.

### 27 Continuity and compactness

Suppose that $X$ and $Y$ are topological spaces, and that $f$ is a continuous mapping from $X$ to $Y$. If $E \subseteq X$ is compact, then

$$f(E) = \{ y \in Y : y = f(x) \text{ for some } x \in E \}$$

is compact in $Y$. To see this, let $\{V_i\}_{i \in I}$ be any open covering of $f(E)$ in $Y$. For every $i \in I$, put

$$U_i = f^{-1}(V_i) = \{ x \in X : f(x) \in V_i \}.$$
Because \( f : X \to Y \) is continuous, \( U_i \) is an open set in \( X \) for every \( i \in I \). The compactness of \( E \) in \( X \) implies that there are finitely many indices \( i_1, \ldots, i_n \in I \) such that
\[
E \subseteq U_{i_1} \cup \cdots \cup U_{i_n}.
\]
This implies in turn that
\[
f(E) \subseteq V_{i_1} \cup \cdots \cup V_{i_n}.
\]
This uses the observation that \( f(1_{A}) \) is open in \( Y \) for every \( A \subseteq Y \). Hence we get a finite subcovering of \( f(E) \) from the initial open covering \( \{V_i\}_{i \in I} \) in \( Y \), and it follows that \( f(E) \) is compact in \( Y \).

### 27.1 The extreme value theorem

Suppose that \( f \) is a continuous real-valued function on a topological space \( X \), and that \( E \) is a compact subset of \( X \). Thus \( f(E) \) is a compact set in \( \mathbb{R} \), as before. In particular, \( f(E) \) is a closed set in \( \mathbb{R} \), because the real line is Hausdorff with respect to the standard topology. We also know that compact subsets of metric spaces are totally bounded, so that \( f(E) \) is totally bounded with respect to the standard metric on \( \mathbb{R} \). This implies that \( f(E) \) is a bounded set in \( \mathbb{R} \), and it follows that \( f(E) \) is also closed, one can check that the supremum and infimum of \( f(E) \) are contained in \( f(E) \), which is to say that the maximum and minimum of \( f \) on \( E \) are attained.

### 27.2 Semicontinuity

A real-valued function \( f \) on a topological space \( X \) is said to be upper semicontinuous if \( f^{-1}((a, +\infty)) \) is an open set in \( X \) for every \( a \in \mathbb{R} \). Similarly, \( f : X \to \mathbb{R} \) is said to be lower semicontinuous if \( f^{-1}((-\infty, b)) \) is an open set in \( X \) for every \( b \in \mathbb{R} \). Thus continuous real-valued functions are both upper and lower semicontinuous, since \( (-\infty, b) \) and \( (a, +\infty) \) are open subsets of \( \mathbb{R} \) for every \( a, b \in \mathbb{R} \). Conversely, if \( f : X \to \mathbb{R} \) is both upper and lower semicontinuous, then it is easy to see that \( f \) is continuous, because
\[
f^{-1}((a, b)) = f^{-1}((a, +\infty)) \cap f^{-1}((-\infty, b))
\]
is an open set in \( X \) for every \( a, b \in \mathbb{R} \) with \( a < b \).

Suppose that \( f : X \to \mathbb{R} \) is upper semicontinuous and that \( E \subseteq X \) is compact. Of course,
\[
X = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, n)),
\]
and one can use the compactness of $E$ to get that $E \subseteq f^{-1}((-\infty, n))$ for some $n$, so that $f(E)$ has an upper bound in $\mathbb{R}$. If $E \neq \emptyset$, then it follows that $f(E)$ has a supremum in $\mathbb{R}$, and we put $t = \sup f(E)$. Suppose for the sake of a contradiction that $t \not\in f(E)$, so that

$$E \subseteq \bigcup_{n=1}^{\infty} f^{-1}((-\infty, t - 1/n)).$$

This implies that $E \subseteq f^{-1}((-\infty, t - 1/n))$ for some $n$, by compactness. Hence $t - 1/n$ is an upper bound for $f(E)$, contradicting the fact that $t$ is the supremum of $f(E)$. This shows that $f$ attains its maximum on $E$ when $f : X \to \mathbb{R}$ is upper semicontinuous and $E \subseteq X$ is nonempty and compact. Similarly, $f$ attains its minimum on $E$ when $f : X \to \mathbb{R}$ is lower semicontinuous and $E \subseteq X$ is nonempty and compact. This can be derived from the previous argument applied to $-f$, because $f : X \to \mathbb{R}$ is lower semicontinuous if and only if $-f$ is upper semicontinuous.

28 Characterizations of compactness

Let $X$ be a topological space, and suppose that $K \subseteq X$ is compact. If $\{E_i\}_{i \in I}$ is a family of closed subsets of $X$ such that

$$(28.1) \quad K \cap E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_n} \neq \emptyset$$

for every collection $i_1, \ldots, i_n$ of finitely many indices in $I$, then

$$(28.2) \quad K \cap \bigcap_{i \in I} E_i \neq \emptyset.$$ 

Otherwise, the complements $X \setminus E_i$ of the $E_i$’s in $X$ form an open covering of $K$ with no finite subcovering. Conversely, if $K$ has this property, then one can check that $K$ is compact, by considering the complements of the sets in an open covering of $K$.

Suppose that $K \subseteq X$ and that $\{E_i\}_{i \in I}$ is a family of closed subsets of $X$ for which (28.1) holds for any $i_1, \ldots, i_n \in I$. Let $\mathcal{A}$ be the collection of finite nonempty subsets of $I$, ordered by inclusion. For every $A \in \mathcal{A}$, let $x_A$ be an element of $K \cap \bigcap_{i \in A} E_i$, which exists, by hypothesis. Thus $\{x_A\}_{A \in \mathcal{A}}$ is a net of elements of $K$ indexed by $\mathcal{A}$ as a directed system. If there is an $x \in K$ such that for every open set $U \subseteq X$ with $x \in U$ and every $A \in \mathcal{A}$ there is an $A' \in \mathcal{A}$ such that $A \subseteq A'$ and $x_{A'} \in U$, then $x \in E_i$ for every $i \in I$, because $x$ is adherent to $E_i$ for each $i \in I$, and hence (28.2) holds.

Conversely, let $K$ be a compact set in $X$, let $(A, \prec)$ be a directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of $K$. For every $a \in A$, let $E_a$ be the closure in $X$ of the set of $x_b$ with $b \in A$ and $a \prec b$. If $a_1, \ldots, a_n \in A$, then there is an $a \in A$ such that $a_i \prec a$ for $i = 1, \ldots, n$, and thus $E_a \subseteq E_{a_i}$ for each $i$. Hence $K \cap E_{a_1} \cap \cdots \cap E_{a_n}$ contains $K \cap E_a$ and is nonempty in particular.
Because $K$ is compact, there is an $x \in K$ such that $x \in E_a$ for every $a \in A$, and thus for every open set $U \subseteq X$ with $x \in U$ and every $a \in A$ there is a $b \in A$ such that $a < b$ and $x_b \in U$, as desired.

## 28.1 Countable and sequential compactness

A subset $K$ of a topological space $X$ is said to be **countably compact** if every countable open covering of $K$ in $X$ can be reduced to a finite subcovering. In analogy with the previous characterization of compact sets in terms of families of closed subsets of $X$, countable compactness can be characterized in the same way in terms of countable families of closed subsets of $X$. More precisely, $K \subseteq X$ is countably compact if for every sequence $U_1, U_2, U_3, \ldots$ of open subsets of $X$ such that $K \subseteq \bigcup_{j=1}^{\infty} U_j$, we have that $K \subseteq \bigcup_{j=1}^{n} U_j$ for some positive integer $n$. We may as well restrict our attention to the case where $U_j \subseteq U_{j+1}$ for each $j$, since this can be arranged by replacing $U_j$ with $\bigcup_{i=1}^{n} U_j$ for each $j$. This is equivalent to saying that if $E_1, E_2, E_3, \ldots$ is a sequence of closed subsets of $X$ such that $E_{j+1} \subseteq E_j$ and $K \cap E_j \neq \emptyset$ for each $j$, then $K \cap \bigcap_{j=1}^{\infty} E_j \neq \emptyset$, by taking complements in $X$, as before.

If $K \subseteq X$ has the strong limit point property, then we have seen that $K$ is countably compact. Conversely, suppose that $K$ is countably compact, and let $x_1, x_2, x_3, \ldots$ be a sequence of elements of $K$. Let $E_l$ be the closure in $X$ of the set of $x_j$ with $j \geq l$ for each positive integer $l$. By construction, $E_l$ is a closed subset of $X$ such that $E_{l+1} \subseteq E_l$ and $K \cap E_l \neq \emptyset$ for each $l$. Thus countable compactness of $K$ implies that there is an element $p$ of $K \cap \bigcap_{l=1}^{\infty} E_l$, as in the previous paragraph. If $U \subseteq X$ is an open set with $p \in U$, then for each $l \in \mathbb{Z}^+$ there is a $j \geq l$ such that $x_\ell \in U$, because $p \in E_l$. In particular, if the $x_\ell$’s are distinct elements of $K$, then this implies that $p$ is a strong limit point of the set of $x_\ell$’s. It follows that $K$ has the strong limit point property when $K$ is countably compact.

A set $K \subseteq X$ is said to be **sequentially compact** if for every sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $K$ there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ and an $x \in K$ such that $\{x_{n_i}\}_{i=1}^{\infty}$ converges to $x$. If $K$ is sequentially compact, then it is easy to see that $K$ has the strong limit point property. More precisely, if $A \subseteq K$ has infinitely many elements, then one can apply sequential compactness to a sequence of distinct elements of $A$. This leads to a limit in $K$ of a convergent subsequence of this sequence, which is a strong limit point of $A$. Conversely, if $X$ satisfies the local countability condition at every point, and if $K \subseteq X$ has the strong limit point property, then $K$ is sequentially compact. To see this, let $\{x_i\}_{i=1}^{\infty}$ be a sequence of elements of $K$. If there is an $x \in K$ such that $x = x_i$ for infinitely many positive integers $i$, then $\{x_i\}_{i=1}^{\infty}$ has a constant subsequence that converges to $x$ trivially. Otherwise, the set $A \subseteq K$ consisting of the $x_i$’s has infinitely many elements, and there is a strong limit point $x \in K$ of $A$. Using the local countability condition for $X$ at $x$, one can show that there is a subsequence of $\{x_i\}_{i=1}^{\infty}$ that converges to $x$, as desired.
28.2 Some variants

Let \((M, d(x, y))\) be a metric space. If \(\{x_i\}_{i=1}^{\infty}\) is a Cauchy sequence in \(M\) with a convergent subsequence \(\{x_{i_l}\}_{l=1}^{\infty}\), then \(\{x_i\}_{i=1}^{\infty}\) converges to the same limit. Hence every Cauchy sequence in a sequentially compact set in \(M\) converges.

Let us say that \(E \subseteq M\) has property (*) if for every set \(A \subseteq E\) with infinitely many elements and every \(\epsilon > 0\) there is a bounded set \(B \subseteq A\) with infinitely many elements such that \(\text{diam}\, B < \epsilon\). Equivalently, \(E\) has property (\(*\)) if for every sequence of elements of \(E\) and every \(\epsilon > 0\) there is a subsequence of the sequence whose terms are contained in a bounded set with diameter less than \(\epsilon\).

If \(E\) is totally bounded, then \(E\) has property (\(*\)), and the converse holds by the same argument used to show that a set with the limit point property is totally bounded. If every sequence in \(E\) has a subsequence which is a Cauchy sequence, then \(E\) has property (\(*\)). Conversely, suppose that \(E \subseteq M\) has property (\(*\)), and that a sequence of elements of \(E\) is given. Using property (\(*\)), one can get a subsequence of the sequence whose terms are contained in a bounded set with diameter less than \(1/2\). For each positive integer \(n\), one can apply property (\(*\)) to get a subsequence of the initial sequence which is also a subsequence of the previous subsequence when \(n \geq 2\) whose terms are contained in a bounded set with diameter less than \(1/n\). The sequence whose \(n\)th term is the \(n\)th term of the \(n\)th subsequence obtained in this way is a subsequence of the initial sequence and is a Cauchy sequence. Therefore, \(E \subseteq M\) is totally bounded if and only if every sequence of elements of \(E\) has a subsequence which is a Cauchy sequence. If \(M\) is complete, then it follows that \(E \subseteq M\) is sequentially compact if and only if \(E\) is closed in \(M\) and totally bounded. One can also show more directly that a subset of a complete metric space is compact when it is closed and totally bounded, using an argument like the traditional proof of compactness of closed intervals in the real line.

29 Products of compact sets

Let \(X\) be a topological space, and suppose that \(B\) is a collection of open subsets of \(X\) which forms a base for the topology of \(X\). If \(K \subseteq X\) is compact with respect to \(B\), in the sense that every covering of \(K\) by elements of \(B\) can be reduced to a finite subcovering, then \(K\) is compact in the usual sense. For let \(\{U_i\}_{i \in I}\) be an arbitrary open covering of \(K\) in \(X\), and let \(B'\) be the set of \(V \in B\) for which there is an \(i \in I\) such that \(V \subseteq U_i\). Because \(B\) is a base for the topology of \(X\), each \(U_i\) is equal to the union of the \(V \in B\) such that \(V \subseteq U_i\). Therefore \(K \subseteq \bigcup_{i \in I} U_i = \bigcup_{V \in B'} V\), which is to say that the \(V\)'s in \(B'\) cover \(K\). By hypothesis, there are \(V_1, \ldots, V_n \in B'\) which cover \(K\). The definition of \(B'\) implies that there are \(i_1, \ldots, i_n \in I\) such that \(V_l \subseteq U_{i_l}\) for \(l = 1, \ldots, n\), and hence that \(U_{i_1}, \ldots, U_{i_n}\) cover \(K\). This shows that \(K\) is compact. There is an analogous statement for the Lindelöf property, and this argument is very similar.
to the one used to show that every $E \subseteq X$ has the Lindelöf property when there is a countable base for the topology of $X$.

Now let $X_1$, $X_2$ be topological spaces, and consider the Cartesian product $X_1 \times X_2$ with the product topology. If $K_1$, $K_2$ are compact subsets of $X_1$, $X_2$, respectively, then $K_1 \times K_2$ is compact in $X_1 \times X_2$. To show this, it is enough to show that every covering $\{U_i \times V_i\}_{i \in I}$ of $K_1 \times K_2$ by products of open sets in $X_1$, $X_2$ can be reduced to a finite subcovering, by the remarks in the preceding paragraph. For every $x \in K_1$, let $I(x)$ be the set of $i \in I$ such that $x \in U_i$. Since $\{x\} \times K_2 \subseteq K_1 \times K_2$, $\{V_i\}_{i \in I(x)}$ is an open covering of $K_2$ in $X_2$. The compactness of $K_2$ in $X_2$ implies that there is a finite set $I_0(x) \subseteq I(x)$ such that $K_2$ is covered by $V_i, i \in I_0(x)$. If $U(x)$ is the intersection of $U_i, i \in I_0(x)$, then $U(x)$ is an open set in $X_1$ which contains $x$. The compactness of $K_1$ implies that there are $x_1, \ldots, x_n \in K_1$ such that $K_1$ is covered by the $U(x_i)$'s, $1 \leq i \leq n$. Because $U(x) \times K_2$ is covered by $U_i \times V_i, i \in I_0(x)$, for every $x \in K_1$, it follows that $K_1 \times K_2$ is covered by the $U_i \times V_i$'s with $i \in I_0(x_1) \cup \cdots \cup I_0(x_n)$, as desired.

### 29.1 Sequences of subsequences

Let $X_1, X_2, \ldots$ be a sequence of topological spaces, and let $X = \prod_{i=1}^{\infty} X_i$ be their Cartesian product. Suppose that $E_i \subseteq X_i, i \geq 1$, are sequentially compact, and consider their Cartesian product $E = \prod_{i=1}^{\infty} E_i$. We would like to show that $E$ is sequentially compact in $X$ when $X$ is equipped with the product topology. It follows from the definition of the product topology that a sequence of elements of $X$ converges to an element of $X$ if and only if the corresponding sequence of coordinates in $X_1$ converges to the $i$th coordinate of the limit for every $i \geq 1$. Suppose that a sequence of elements of $E$ is given. Because $E_1$ is sequentially compact, there is a subsequence of this sequence whose coordinates in $X_1$ converge to an element of $E_1$. Similarly, because $E_2$ is sequentially compact, there is a subsequence of the subsequence for which the coordinates in $X_2$ converge to an element of $E_2$. Proceeding in this way, we get a sequence of subsequences of the initial sequence, where each new subsequence is a subsequence of the preceding subsequence, and where the coordinates in $X_n$ of the $n$th subsequence converge to an element of $E_n$ for every $n \geq 1$. If we form a sequence in which the $n$th term is equal to the $n$th term of the $n$th subsequence, then this sequence is a subsequence of the initial sequence which converges in $X$ to an element of $E$. Of course, the analogous argument for finite products is simpler, and in particular the last step would not be needed in that case.

### 29.2 Compactness of closed intervals

Let $B$ be the set of all binary sequences, i.e., the set of sequences $x = \{x_i\}_{i=1}^{\infty}$ such that $x_i = 0$ or 1 for each $i$. Equivalently, $B$ is the Cartesian product $\prod_{i=1}^{\infty} B_i$ with $B_i = \{0, 1\}$ for each $i$. Using the product topology, based on the discrete topology on the $B_i$'s, $B$ becomes a Hausdorff topological space which is sequentially compact, since the $B_i$'s are sequentially compact trivially. It follows that $B$ is a compact space, using the fact that the topology on $B$ is determined
by a metric, or by observing that $\mathcal{B}$ has a countable base for its topology and therefore has the Lindelöf property.

Consider the real-valued function $f$ on $\mathcal{B}$ defined by $f(x) = \sum_{i=1}^{\infty} x_i 2^{-i}$, which associates to each binary sequence a real number in the usual way. One can check that $f : \mathcal{B} \rightarrow \mathbb{R}$ is a continuous mapping with respect to the product topology on $\mathcal{B}$. It follows that the closed unit interval $[0, 1]$ is a compact set in the real line, since $f(\mathcal{B}) = [0, 1]$. Of course, one can also show this more directly.

### 30 Filters

Let $X$ be a set. A filter on $X$ is a nonempty collection $\mathcal{F}$ of nonempty subsets of $X$ such that

(30.1) $A \cap B \in \mathcal{F}$ for every $A, B \in \mathcal{F}$

and

(30.2) if $A \in \mathcal{F}$ and $A \subseteq E \subseteq X$, then $E \in \mathcal{F}$.

In particular, this implies that $X \in \mathcal{F}$ for every filter $\mathcal{F}$ on $X$. Conversely, $\mathcal{F} = \{X\}$ is a filter on any nonempty set $X$.

Similarly, if $X$ is an infinite set, then

(30.3) $\mathcal{F} = \{A \subseteq X : X \setminus A$ has only finitely many elements$\}$

is a filter on $X$. If $X$ is any set and $p \in X$, then it is easy to see that

(30.4) $\mathcal{F}_p = \{A \subseteq X : p \in A\}$

is a filter on $X$. More generally, if $A$ is a nonempty subset of any set $X$, then

(30.5) $\mathcal{F}_A = \{B \subseteq X : A \subseteq B\}$

is a filter on $A$.

Suppose now that $X$ is a topological space. A filter $\mathcal{F}$ on $X$ is said to converge to a point $p \in X$ if for every open set $U$ in $X$ such that $p \in U$, we have that $U \in \mathcal{F}$. Thus for instance the filter $\mathcal{F}_p$ in (30.4) automatically converges to $p$, for any topological space $X$. Conversely, if a filter $\mathcal{F}$ on $X$ converges to $p \in X$ and $X$ is equipped with the discrete topology, then $U = \{p\}$ is an open set in $X$ that contains $p$, and hence $\{p\} \in \mathcal{F}$. This implies that

(30.6) $\mathcal{F}_p \subseteq \mathcal{F}$,

because of (30.2). In the other direction, if $A \in \mathcal{F}$, then $A \cap \{p\} \in \mathcal{F}$, by (30.1), so that $A \cap \{p\} \neq \emptyset$, and hence $p \in A$. Thus $\mathcal{F} \subseteq \mathcal{F}_p$, which implies that

(30.7) $\mathcal{F} = \mathcal{F}_p$

under these conditions.
Let $X$ be any topological space again, let $p$ be an element of $X$, and put
\[(30.8) \quad \mathcal{F}(p) = \{ A \subseteq X : p \in U \subseteq A \text{ for some open set } U \subseteq X \}.
\]
It is easy to see that this is a filter on $X$, because the intersection of two open subsets of $X$ is also an open set in $X$. Note that $\mathcal{F}(p)$ converges to $p$ by construction, and that a filter $\mathcal{F}$ on $X$ converges to $p$ if and only if $\mathcal{F}(p) \subseteq \mathcal{F}$.

If $X$ is a Hausdorff topological space, then the limit of a convergent filter on $X$ is unique. To see this, suppose for the sake of a contradiction that $\mathcal{F}$ is a filter on $X$ that converges to $p, q \in X$, and that $p \neq q$. If $X$ is Hausdorff, then there are disjoint open subsets $U, V$ of $X$ such that $p \in U$ and $q \in V$. The convergence of $\mathcal{F}$ to $p$ and to $q$ implies that $U$ and $V$ are both elements of $\mathcal{F}$, and hence that $U \cap V \in \mathcal{F}$. This is a contradiction, since $U \cap V = \emptyset$, while the elements of $\mathcal{F}$ are supposed to be nonempty subsets of $X$.

Conversely, if $X$ is not Hausdorff, then there is a filter on $X$ that converges to two different elements of $X$. More precisely, if $X$ is not Hausdorff, then there are elements $p, q$ of $X$ such that $U \cap V \neq \emptyset$ for every pair of open subsets $U, V$ of $X$ with $p \in U, q \in V$. In this case, one can check that
\[(30.9) \quad \mathcal{F} = \{ A \subseteq X : U \cap V \subseteq A \text{ for some open sets } U, V \subseteq X \text{ such that } p \in U \text{ and } q \in V \}
\]
is a filter on $X$ that converges to both $p$ and $q$, as desired.

If $\mathcal{F}$ is a filter on a topological space $X$ that converges to a point $p \in X$, and if $E \in \mathcal{F}$, then $p$ is adherent to $E$, so that $p \in E$. To see this, let $U$ be an open set in $X$ such that $p \in U$. Because $\mathcal{F}$ converges to $p$, we have that $U \in \mathcal{F}$, and hence $E \cap U \in \mathcal{F}$, by (30.2). This implies that $E \cap U \neq \emptyset$, by the definition of a filter, as desired.

Conversely, if $p \in X$ is adherent to a set $E \subseteq X$, then there is a filter $\mathcal{F}$ on $X$ that converges to $p$ and contains $E$ as an element. By hypothesis, $E \cap U \neq \emptyset$ for every open set $U \subseteq X$ such that $p \in U$, and we can take
\[(30.10) \quad \mathcal{F} = \{ A \subseteq X : E \cap U \subseteq A \text{ for some open set } U \subseteq X \text{ with } p \in U \}.
\]
It is not difficult to verify that this has the required properties.

30.1 Nets and filters
Let $A$ be a nonempty set, and let $\prec$ be a partial ordering on $A$ such that $(A, \prec)$ is a directed system. Also let $\{ x_a \}_{a \in A}$ be a net of elements of a set $X$, and put
\[(30.11) \quad E_a = \{ x_b : b \in A, a \prec b \}
\]
for each $a \in A$. Thus
\[(30.12) \quad \mathcal{F} = \{ E \subseteq X : E_a \subseteq E \text{ for some } a \in A \}
\]
is a nonempty collection of nonempty subsets of $X$, and we would like to check that $\mathcal{F}$ is a filter on $X$. Let $E, E' \in \mathcal{F}$ be given, and let $a, a'$ be elements of
A such that $E_a \subseteq E$ and $E_a' \subseteq E'$. Because $A$ is a directed system, there is a $b \in A$ such that $a, a' \prec b$, which implies that

$$E_b \subseteq E_a \cap E_a' \subseteq E \cap E'. \tag{30.13}$$

Hence $E \cap E' \in \mathcal{F}$, so that $\mathcal{F}$ satisfies (30.1). By construction, $\mathcal{F}$ also satisfies (30.2), and is therefore a filter.

Suppose now that $X$ is a topological space. If $\{x_a\}_{a \in A}$ converges to a point $x \in X$, then for each open set $U$ in $X$ with $x \in U$ there is an $a \in A$ such that $E_a \subseteq U$. This implies that $U \in \mathcal{F}$, so that $\mathcal{F}$ also converges to $x$. Conversely, if $\mathcal{F}$ converges to $x$, then $U \in \mathcal{F}$ for every open set $U \subseteq X$ with $x \in U$. This implies that $E_a \subseteq U$ for some $a \in A$, and hence that $\{x_a\}_{a \in A}$ converges to $x$.

In the other direction, let $\mathcal{F}$ be a filter on a set $X$, and let us consider some nets associated to $\mathcal{F}$, as follows. Let $\prec$ be the partial ordering on $\mathcal{F}$ corresponding to reversed inclusion, so that $E_1 \prec E_2$ when $E_1, E_2 \in \mathcal{F}$ and $E_2 \subseteq E_1$. If $E, E' \in \mathcal{F}$, then $E \cap E' \in \mathcal{F}$ and $E, E' \prec E \cap E'$, which shows that $\mathcal{F}$ is a directed system with respect to $\prec$. If $x(E) \in E$ for each $E \in \mathcal{F}$, then $\{x(E)\}_{E \in \mathcal{F}}$ is a net of elements of $X$ indexed by $\mathcal{F}$ as a directed system. If $X$ is a topological space and $\mathcal{F}$ converges to a point $x$ in $X$, then it is easy to see that each of these nets associated to $\mathcal{F}$ also converges to $x$.

Conversely, suppose that $\mathcal{F}$ does not converge to $x$ in $X$. This means that there is an open set $U$ in $X$ such that $x \in U$ and $U \notin \mathcal{F}$. If $E \in \mathcal{F}$, then $E \not\subseteq U$, since otherwise $U$ would be an element of $\mathcal{F}$, by the definition of a filter. If $x(E) \in E \setminus U$ for each $E \in \mathcal{F}$, then $\{x(E)\}_{E \in \mathcal{F}}$ is a net associated to $\mathcal{F}$ as in the previous paragraph that does not converge to $x$.

### 30.2 Mappings and filters

Let $X$ and $Y$ be sets, let $f$ be a mapping from $X$ into $Y$, and let $\mathcal{F}$ be a filter on $X$. Under these conditions, it is easy to see that

$$f_*(\mathcal{F}) = \{E \subseteq Y : f^{-1}(E) \in \mathcal{F}\} \tag{30.14}$$

is a filter on $Y$.

Now let $X$, $Y$ be topological spaces. If $\mathcal{F}$ converges to $p \in X$ and $f$ is continuous at $p$, then we would like to check that $f_*(\mathcal{F})$ converges to $f(p)$. Let $V$ be an open set in $Y$ that contains $f(p)$ as an element. If $f$ is continuous at $p$, then there is an open set $U$ in $X$ such that $p \in U$ and $U \subseteq f^{-1}(V)$. If $\mathcal{F}$ converges to $p$ in $X$, then $U \in \mathcal{F}$, and hence $V \in f_*(\mathcal{F})$, as desired.

Conversely, let $\mathcal{F}(p)$ be the filter on $X$ generated by open sets $U \subseteq X$ such that $p \in U$, as in (30.8). Suppose that $f_*(\mathcal{F}(p))$ converges to $f(p)$ in $Y$, and let us check that $f$ is continuous at $p$. If $V$ is an open set in $Y$ that contains $f(p)$, then $V \in f_*(\mathcal{F}(p))$ by hypothesis, and hence $f^{-1}(V) \in \mathcal{F}(p)$. This implies that there is an open set $U$ in $X$ such that $p \in U \subseteq f^{-1}(V)$, as desired.
31 Refinements

Let $X$ be a set, and let $\mathcal{F}$ be a filter on $X$. A filter $\mathcal{F}'$ is said to be a refinement of $\mathcal{F}$ when

$$\mathcal{F} \subseteq \mathcal{F'}.$$  

Suppose now that $X$ is a topological space, $\mathcal{F}$ is a filter on $X$, $p \in X$, and

$$p \in \overline{E}$$

for each $E \in \mathcal{F}$. This last is equivalent to the condition that

$$E \cap U \neq \emptyset$$

for every open set $U \subseteq X$ such that $p \in U$. It is easy to see that

$$\mathcal{F}' = \{ A \subseteq X : E \cap U \subseteq A \text{ for some } E \in \mathcal{F} \text{ and open set } U \subseteq X \text{ such that } p \in U \}$$

is a filter on $X$, because $\mathcal{F}$ is a filter and the intersection of two open subsets of $X$ is also open. If $A \in \mathcal{F}$, then $A \in \mathcal{F}'$, since we can take $E = A$ and $U = X$ in (31.4), and hence $\mathcal{F}'$ is a refinement of $\mathcal{F}$. Similarly, if $U$ is an open set in $X$ that contains $p$, then $U \in \mathcal{F}'$, since we can take $E = X$ in (31.4), and thus $\mathcal{F}'$ converges to $p$ in $X$.

Conversely, suppose that $\mathcal{F}$ is a filter on $X$, and that $\mathcal{F}'$ is a filter on $X$ which is a refinement of $\mathcal{F}$ and which converges to a point $p \in X$. The latter implies that $U \in \mathcal{F}'$ for every open set $U$ in $X$ that contains $p$. If $E \in \mathcal{F}$, then $E \in \mathcal{F}'$, because $\mathcal{F}'$ is a refinement of $\mathcal{F}$, and hence

$$E \cap U \in \mathcal{F}'.$$

In particular, $E \cap U \neq \emptyset$ for every $E \in \mathcal{F}$ and open set $U \subseteq X$ that contains $p$. It follows that $p \in \overline{E}$ for each $E \in \mathcal{F}$, as in the previous paragraph.

31.1 Another characterization of compactness

Let $X$ be a topological space, and suppose that $K \subseteq X$ is compact. Also let $\mathcal{F}$ be a filter on $X$ such that $K \in \mathcal{F}$, and consider the collection of closed sets $\overline{E}$ in $X$, with $E \in \mathcal{F}$. If $E_1, \ldots, E_n$ are finitely many elements of $\mathcal{F}$, then

$$E_1 \cap \cdots \cap E_n \cap K \in \mathcal{F},$$

and in particular this set is not empty. It follows that

$$\overline{E_1 \cap \cdots \cap E_n} \cap K \neq \emptyset,$$

and hence that

$$\left( \bigcap_{E \in \mathcal{F}} \overline{E} \right) \cap K \neq \emptyset,$$
by an earlier characterization of compactness. This implies that there is filter \( F' \) on \( X \) which is a refinement of \( F \) and which converges to a point \( p \in K \), by the previous discussion of refinements.

Conversely, suppose that \( K \) is a subset of \( X \) with the property that any filter \( F \) on \( X \) with \( K \in \mathcal{F} \) has a refinement that converges to an element of \( K \), and let us show that \( K \) is compact. Let \( \{E_i\}_{i \in I} \) be a family of closed subsets of \( X \) such that

\[
E_{i_1} \cap \cdots \cap E_{i_n} \cap K \neq \emptyset
\]

for every collection \( i_1, \ldots, i_n \) of finitely many elements of \( I \). If

\[
\mathcal{F} = \{A \subseteq X : E_{i_1} \cap \cdots \cap E_{i_n} \cap K \subseteq A \text{ for some } i_1, \ldots, i_n \in I\},
\]

then it is easy to check that \( \mathcal{F} \) is a filter on \( X \) that contains \( K \) as an element. By hypothesis, there is a refinement \( \mathcal{F}' \) of \( \mathcal{F} \) that converges to an element \( p \) of \( K \). This implies that \( p \in \mathcal{F} \) for each \( E \in \mathcal{F} \), by the previous discussion of refinements again. Of course, \( E_i \in \mathcal{F} \) for each \( i \in I \), by construction, and so \( p \in E_i \) for each \( i \in I \), because \( E_i \) is a closed set in \( X \) for every \( i \in I \). Thus

\[
p \in \left( \bigcap_{i \in I} E_i \right) \cap K,
\]

and hence

\[
\left( \bigcap_{i \in I} E_i \right) \cap K \neq \emptyset.
\]

This implies that \( K \) is a compact set in \( X \), by an earlier characterizations of compactness.

### 32 Ultrafilters

Let \( X \) be a set, and let \( \mathcal{F} \) be a filter on \( X \). We say that \( \mathcal{F} \) is an ultrafilter on \( X \) if it is maximal in the sense that if \( \mathcal{F}' \) is a filter on \( X \) which is a refinement of \( \mathcal{F} \), then \( \mathcal{F}' = \mathcal{F} \). For example, if \( p \in X \), then it is easy to see that the collection \( \mathcal{F}_p \) of subsets of \( X \) that contain \( p \) as an element is an ultrafilter on \( X \). Ultrafilters of this form are said to be principal ultrafilters on \( X \).

Suppose that \( \mathcal{F} \) is a filter on \( X \), \( A \subseteq X \), and

\[
A \cap E \neq \emptyset
\]

for every \( E \in \mathcal{F} \). In this case, it is easy to see that

\[
\mathcal{F}^A = \{B \subseteq X : A \cap E \subseteq B \text{ for some } E \in \mathcal{F}\}
\]

is a filter on \( X \) which is a refinement of \( \mathcal{F} \). If \( \mathcal{F} \) is an ultrafilter on \( X \), then it follows that \( \mathcal{F}^A = \mathcal{F} \) under these conditions. Note that \( A \in \mathcal{F}^A \) automatically, so that \( \mathcal{F}^A = \mathcal{F} \) implies that \( A \in \mathcal{F} \). Conversely, if \( A \in \mathcal{F} \), then \( \mathcal{F}^A = \mathcal{F} \).

If \( \mathcal{F} \) is any filter on \( X \), \( A \subseteq X \), and there is an \( E \in \mathcal{F} \) such that \( A \cap E = \emptyset \), then \( E \subseteq X \setminus A \), and therefore \( X \setminus A \in \mathcal{F} \). This shows that either \( A \cap E \neq \emptyset \) and
for every $E \in \mathcal{F}$, or $X \setminus A \in \mathcal{F}$. If $\mathcal{F}$ is an ultrafilter, then we have seen that $A \in \mathcal{F}$ in the first case, as in the preceding paragraph.

Now suppose that $\mathcal{F}, \mathcal{F}'$ are filters on $X$, and that $\mathcal{F}'$ is a refinement of $\mathcal{F}$. If $A \in \mathcal{F}'$ and $E \in \mathcal{F}$, then $A \cap E \in \mathcal{F}'$, and thus $A \cap E \neq \emptyset$. If $\mathcal{F}$ has the property that $A \in \mathcal{F}$ whenever $A \subseteq X$ satisfies $A \cap E \neq \emptyset$ for every $E \in \mathcal{F}$, then it follows that $\mathcal{F} = \mathcal{F}'$, and hence that $\mathcal{F}$ is an ultrafilter on $X$.

Let $\mathcal{F}$ be a filter on $X$, and suppose that for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. If $A \subseteq X$ satisfies $A \cap E \neq \emptyset$ for every $E \in \mathcal{F}$, then we cannot have $X \setminus A \in \mathcal{F}$, and so we conclude that $A \in \mathcal{F}$. Thus $\mathcal{F}$ satisfies the criterion in the previous paragraph, so that $\mathcal{F}$ is an ultrafilter on $X$. To summarize, a filter $\mathcal{F}$ on $X$ is an ultrafilter if and only if for each $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

### 32.1 Connections with set theory

Using the axiom of choice, through the Hausdorff maximality principle or Zorn’s lemma, one can show that every filter $\mathcal{F}$ on a set $X$ has a refinement $\mathcal{U}$ which is an ultrafilter on $X$. More precisely, the collection of all filters on $X$ is a partially-ordered set with respect to inclusion, which is the ordering corresponding to refinement of filters. Let $\mathcal{C}$ be a nonempty chain of filters on $X$, which is to say a nonempty collection of filters on $X$ such that for every $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}$, either $\mathcal{F}_2$ is a refinement of $\mathcal{F}_1$, or the other way around. Let $\hat{\mathcal{F}}$ be the union of the filters in $\mathcal{C}$, which means that $E \subseteq X$ is an element of $\hat{\mathcal{F}}$ if and only if $E \in \mathcal{F}$ for some $\mathcal{F} \in \mathcal{C}$. It is not difficult to check that $\hat{\mathcal{F}}$ is a filter on $X$ under these conditions. The main point is that if $E_1, E_2 \in \hat{\mathcal{F}}$, then $E_1 \cap E_2 \in \hat{\mathcal{F}}$ too. To see this, let $\mathcal{F}_1, \mathcal{F}_2$ be elements of $\mathcal{C}$ that contain $E_1, E_2$ as elements, respectively. Because $\mathcal{C}$ is a chain of filters on $X$, either $\mathcal{F}_1 \subseteq \mathcal{F}_2$ or $\mathcal{F}_2 \subseteq \mathcal{F}_1$. This implies that $E_1$ and $E_2$ are both contained in $\mathcal{F}_1$, or that they are both contained in $\mathcal{F}_2$. Hence $E_1 \cap E_2$ is an element of $\mathcal{F}_1$ or $\mathcal{F}_2$, by the definition of a filter. In both cases, it follows that $E_1 \cap E_2 \in \hat{\mathcal{F}}$, as desired. Of course, $\hat{\mathcal{F}}$ is a refinement of every element of $\mathcal{C}$, in addition to being a filter on $X$. This permits one to apply the Hausdorff maximality principle or Zorn’s lemma to the partially-ordered set of filters on $X$ that are refinements of a given filter $\mathcal{F}$ on $X$ to show that $\hat{\mathcal{F}}$ is a refinement which is an ultrafilter on $X$.

If $X$ is a topological space and $K \subseteq X$, then we have seen that $K$ is compact if and only if every filter $\mathcal{F}$ on $X$ with $K \in \mathcal{F}$ has a refinement which converges to an element of $K$. If $\mathcal{U}$ is an ultrafilter on $X$, $K \subseteq X$ is compact, and $K \in \mathcal{U}$, then it follows that $\mathcal{U}$ converges to an element of $K$. Conversely, suppose that $K \subseteq X$ and every ultrafilter $\mathcal{U}$ on $X$ with $K \in \mathcal{U}$ converges to an element of $K$. If $\mathcal{F}$ is a filter on $X$ with $K \in \mathcal{F}$, then there is an ultrafilter $\mathcal{U}$ on $X$ which is a refinement of $\mathcal{F}$, as in the previous paragraph. Of course, $K \in \mathcal{U}$, so that $\mathcal{U}$ converges to an element of $K$ by hypothesis, and hence $K$ is compact.

### 32.2 Tychonoff’s theorem

Let $X$ and $Y$ be sets, and let $f$ be a mapping from $X$ into $Y$. If $\mathcal{F}$ is an ultrafilter on $X$, then $f_*(\mathcal{F})$ is an ultrafilter on $Y$. To see this, it suffices to check that for
each $A \subseteq Y$, either $A \in f_*(\mathcal{F})$ or $Y \setminus A \in f_*(\mathcal{F})$. This is the same as saying that either $f^{-1}(A) \in \mathcal{F}$ or $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in \mathcal{F}$, which holds because $\mathcal{F}$ is an ultrafilter on $X$.

Now let $I$ be a nonempty set, and suppose that $X_i$ is a topological space for each $i \in I$. Also let $K_i$ be a compact subset of $X_i$ for each $i \in I$. Under these conditions, Tychonoff’s theorem states that $K = \prod_{i \in I} K_i$ is a compact subset of $X = \prod_{i \in I} X_i$ with respect to the product topology. To prove this, it is enough to show that if $\mathcal{U}$ is an ultrafilter on $X$ such that $K \in \mathcal{U}$, then $\mathcal{U}$ converges on $X$ to an element of $K$. Let $p_i : X \to X_i$ be the standard coordinate projection for each $i \in I$, which sends elements of $X$ to their $i$th coordinates in $X_i$. The argument in the preceding paragraph implies that $(p_i)_*(\mathcal{U})$ is an ultrafilter on $X_i$ for each $i \in I$. It is also easy to see that $K_i \in (p_i)_*(\mathcal{U})$ for each $i \in I$, because $K \in \mathcal{U}$. It follows that $(p_i)_*(\mathcal{U})$ converges in $X_i$ to an element $x_i$ of $K_i$ for each $i \in I$, because $K_i$ is a compact subset of $X_i$ for each $i$. If $x$ is the element of $K$ such that $p_i(x) = x_i$ for every $i \in I$, then one can check that $\mathcal{U}$ converges to $x$ on $X$, as desired.

### 33 Continuous real-valued functions

If $X$, $Y$ are topological spaces, then the space of continuous mappings from $X$ to $Y$ is denoted $\mathcal{C}(X,Y)$. In the special case where $Y$ is the real line with the standard topology, we may use the abbreviation $\mathcal{C}(X)$ instead of $\mathcal{C}(X,\mathbb{R})$.

Suppose that $X$ is a topological space, $p \in X$, and $f_1$, $f_2$ are real-valued functions on $X$ which are continuous at $p$, and let us check that the sum $f_1 + f_2$ is continuous at $p$ too. Let $\epsilon > 0$ be given, and let $U_1$, $U_2$ be open subsets of $X$ such that $p \in U_1, U_2$,

\[ |f_1(x) - f_1(p)| < \frac{\epsilon}{2} \]

for every $x \in U_1$, and

\[ |f_2(x) - f_2(y)| < \frac{\epsilon}{2} \]

for every $x \in U_2$. By the triangle inequality for the absolute value function,

\[ |(f_1(x) + f_2(x)) - (f_1(p) + f_2(p))| \leq |f_1(x) - f_1(p)| + |f_2(x) - f_2(p)| < \epsilon \]

for every $x \in U_1 \cap U_2$. It follows that $f_1 + f_2$ is continuous at $p$, since $U_1 \cap U_2$ is an open set in $X$ that contains $p$. Similarly, one can show that the product $f_1 f_2$ is continuous at $p$. It is a bit simpler to start with the special cases where $f_1$ or $f_2$ is a constant function, and where $f_1(p) = 0$ or $f_2(p) = 0$. The general case can then be derived from these two special cases using the previous assertion for sums. Of course, these arguments are analogous to more classical versions for functions on the real line. As a consequence, $\mathcal{C}(X)$ is a commutative ring with respect to pointwise addition and multiplication, since the sum and product of two continuous real-valued functions on $X$ is a continuous function on $X$.

One can also look at continuity of sums and products in terms of compositions, which is discussed next. More precisely, the continuity of real-valued
functions $f_1$, $f_2$ on $X$ is equivalent to the continuity of the combined mapping

$$F(x) = (f_1(x), f_2(x)),$$

as a mapping from $X$ into $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}^2$ is equipped with the product topology associated to the standard topology on $\mathbb{R}$. Addition and multiplication on $\mathbb{R}$ correspond to mappings from $\mathbb{R}^2$ into $\mathbb{R}$, and one can verify that these mappings are continuous with respect to the standard topology on $\mathbb{R}$ and the associated product topology on $\mathbb{R}^2$, by classical arguments. The sum and product of $f_1$, $f_2$ on $X$ are the same as the compositions of $F$ with these two mappings from $\mathbb{R}^2$ into $\mathbb{R}$, so that their continuity can be derived from the continuity of compositions of continuous mappings.

If $f$ is a real-valued function on $X$ such that $f(x) \neq 0$ for every $x \in X$, then $1/f(x)$ defines a real-valued function on $X$ as well. If $f$ is continuous at a point $p \in X$, then one can show that $1/f$ is continuous at $p$ too, using arguments like those for functions on the real line, as before. Alternatively, $t \mapsto 1/t$ defines a continuous mapping from $\mathbb{R} \setminus \{0\}$ into itself, with respect to the topology induced on $\mathbb{R} \setminus \{0\}$ by the standard topology on $\mathbb{R}$, as in the classical situation. This permits the continuity of $1/f$ on $X$ to be derived from the continuity of compositions of continuous mappings, since $1/f$ can be expressed as the composition of $f$ with $t \mapsto 1/t$ on $\mathbb{R} \setminus \{0\}$.

34 Compositions and inverses

Suppose that $X_1$, $X_2$, and $X_3$ are topological spaces, $f_1$ is a mapping from $X_1$ to $X_2$, and $f_2$ is a mapping from $X_2$ to $X_3$. As usual, the composition $f_2 \circ f_1$ is the mapping from $X_1$ to $X_3$ given by $(f_2 \circ f_1)(x) = f_2(f_1(x))$. If $p \in X_1$, $f_1$ is continuous at $p$, and $f_2$ is continuous at $f_1(p)$, then it is easy to see that $f_2 \circ f_1$ is continuous at $p$. Consequently, $f_1 \in \mathcal{C}(X_1, X_2)$ and $f_2 \in \mathcal{C}(X_2, X_3)$ imply that $f_2 \circ f_1 \in \mathcal{C}(X_1, X_3)$, which one can also check by showing that $(f_2 \circ f_1)^{-1}(W) = f_1^{-1}(f_2^{-1}(W))$ is an open set in $X_1$ whenever $W$ is an open set in $X_3$.

Let $X$, $Y$ be sets, and let $I_X$, $I_Y$ be the identity mappings on $X$ and $Y$, which is to say that $I_X(x) = x$ for every $x \in X$ and $I_Y(y) = y$ for every $y \in Y$. A mapping $f : X \rightarrow Y$ is invertible if there is a mapping $f^{-1} : Y \rightarrow X$ such that $f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_Y$. Because $f^{-1} \circ f = I_X$, $f$ is a one-to-one mapping from $X$ to $Y$, or equivalently $f : X \rightarrow Y$ is an injection, which means that $f(x) = f(x')$ implies $x = x'$ for every $x, x' \in X$. Similarly, because $f \circ f^{-1} = I_Y$, $f$ maps $X$ onto $Y$, or equivalently $f : X \rightarrow Y$ is a surjection, which means that for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Conversely, if $f$ is a one-to-one mapping from $X$ onto $Y$, which is the same as saying that $f$ is a one-to-one correspondence from $X$ onto $Y$ or that $f : X \rightarrow Y$ is a bijection, then $f$ is invertible, and $f^{-1} : Y \rightarrow X$ can be defined by $f^{-1}(y) = x$ when $f(x) = y$ for $x \in X$, $y \in Y$. Observe in particular that the inverse $f^{-1}$ of an invertible mapping $f$ is unique. If $X_1$, $X_2$, and $X_3$ are sets and $f_1 : X_1 \rightarrow X_2$, $f_2 : X_2 \rightarrow X_3$ are invertible mappings, then one can check that $f_2 \circ f_1 : X_1 \rightarrow X_3$ is...
invertible, and that \((f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}\). If \(X, Y\) are topological spaces, then a \textit{homeomorphism} from \(X\) onto \(Y\) is an invertible mapping \(f : X \to Y\) which is continuous and whose inverse \(f^{-1} : Y \to X\) is continuous. If \(X_1, X_2, X_3\) are topological spaces and \(f_1 : X_1 \to X_2, f_2 : X_2 \to X_3\) are homeomorphisms, then \(f_2 \circ f_1 : X_1 \to X_3\) is a homeomorphism too. Of course the inverse of a homeomorphism is a homeomorphism as well.

### 34.1 Compact spaces

Let \(X, Y\) be topological spaces, and suppose that \(f\) is a one-to-one continuous mapping from \(X\) onto \(Y\). If \(X\) is compact and \(Y\) is Hausdorff, then the inverse mapping \(f^{-1}\) is also continuous as a mapping from \(Y\) onto \(X\), which is to say that \(f\) is a homeomorphism from \(X\) onto \(Y\). To see this, it will be convenient to let the inverse mapping be denoted \(g\). It suffices to show that \(g^{-1}(U)\) is an open set in \(Y\) for every open set \(U\) in \(X\). Equivalently, it is enough to show that \(g^{-1}(E)\) is a closed set in \(Y\) for every closed set \(E\) in \(X\). Of course, \(g^{-1}(E) = f(E)\) in this situation. If \(X\) is compact, then every closed set \(E \subseteq X\) is also compact, by previous results. This implies that \(f(E)\) is compact in \(Y\) when \(E \subseteq X\) is closed, because \(f\) is continuous. If \(Y\) is Hausdorff, then compact subsets of \(Y\) are closed, and hence \(g^{-1}(E) = f(E)\) is a closed set in \(Y\) for every closed set \(E \subseteq X\), as desired. It is easy to give examples where \(f^{-1} : Y \to X\) is not continuous when \(X\) is not required to be compact or \(Y\) is not required to be Hausdorff.

### 35 Local compactness

A topological space \(X\) is said to be \textit{locally compact} if for every \(p \in X\) there is an open set \(U \subseteq X\) such that \(p \in U\) and \(U\) is contained in a compact set in \(X\). Suppose that \(X\) is locally compact and that \(K \subseteq X\) is compact. For every \(p \in K\), there is an open set \(U(p) \subseteq X\) such that \(p \in U(p)\) and \(U(p)\) is contained in a compact set. Because \(K\) is compact, there are finitely many elements \(p_1, \ldots, p_n\) of \(K\) such that \(K\) is contained in \(U = \bigcup_{i=1}^n U(p_i)\). Of course \(U\) is an open set in \(X\), and \(U\) is contained in a compact set since the union of finitely many compact sets is a compact set.

If \(X\) is Hausdorff, then \(X\) is locally compact if and only if for every \(p \in X\) there is an open set \(U \subseteq X\) such that \(p \in U\) and \(\overline{U}\) is compact. Suppose that \(X\) is a locally compact Hausdorff topological space, \(p \in X\), and \(U \subseteq X\) is an open set such that \(p \in U\) and \(\overline{U}\) is compact. Observe that \(\partial U = \overline{U} \setminus U\) is a compact set in \(X\), since it is the intersection of the compact set \(\overline{U}\) with the closed set \(X \setminus U\). Because \(p \in U \subseteq X \setminus \partial U\) and \(X\) is Hausdorff, there are disjoint open subsets \(V, W\) of \(X\) such that \(p \in V\) and \(\partial U \subseteq W\). Hence \(U_1 = U \cap V\) is an open set in \(X\), \(p \in U_1\), and \(\overline{U_1} \subseteq U\), which implies that \(\overline{U_1}\) is compact. We also have that \(U_1 \subseteq X \setminus W\) and hence \(\overline{U_1} \subseteq X \setminus W\), since \(W\) is an open set. It follows that \(\overline{U_1} \subseteq U\), since \(\overline{U_1} \subseteq \overline{U}\) and \(\partial U \subseteq W\), by construction. If \(U_0\) is any open set in \(X\) such that \(p \in U_0\), there is an open set \(U \subseteq X\) such that \(p \in U\), \(U \subseteq U_0\), and
$\mathcal{U}$ is compact, i.e., the intersection of $U_0$ with any open set that contains $p$ and has compact closure. By the previous argument, there is an open set $U_1 \subseteq X$ such that $p \in U_1$, $\overline{U_1} \subseteq U \subseteq U_0$, and $\overline{U_1}$ is compact. In particular, a locally compact Hausdorff topological space is regular, since one can take $U_0 = X \setminus E$ when $E \subseteq X$ is closed and $p \notin E$, and then $U_1, X \setminus \overline{U_1}$ are disjoint open subsets of $X$ that contain $p, E$, respectively.

### 36 Localized separation conditions

Let $X$ be a topological space. The Hausdorff property is equivalent to asking that for every $p, q \in X$ with $p \neq q$ there is an open set $U \subseteq X$ such that $p \in U$ and $q$ is not an element of the closure of $U$. Similarly, $X$ is regular if and only if $X$ satisfies the first separation condition and for every $p \in X$ and open set $V \subseteq X$ with $p \in V$ there is an open set $W \subseteq X$ such that $p \in W$ and the closure of $W$ is contained in $V$. Also, normality is the same as saying that $X$ satisfies the first separation condition and for every $A, V \subseteq X$ with $A$ closed, $V$ open, and $A \subseteq V$ there is an open set $W \subseteq X$ such that $A \subseteq W$ and $\overline{W} \subseteq V$. If $X$ is regular and $K, V$ are subsets of $X$ with $K$ compact, $V$ open, and $K \subseteq V$, then there is an open set $W \subseteq X$ such that $K \subseteq W$ and $\overline{W} \subseteq V$, by the usual covering arguments.

Suppose now that $X$ is a locally compact Hausdorff topological space. In this case, $X$ is regular, and every compact set in $X$ is contained in an open set with compact closure. Using the observation at the end of the preceding paragraph, it follows that if $K \subseteq X$ is compact and $V \subseteq X$ is an open set with $K \subseteq V$, there is an open set $W \subseteq X$ such that $K \subseteq W$, $\overline{W} \subseteq V$. If $K_1, K_2$ are disjoint compact subsets of $X$, then there are disjoint open sets $V_1, V_2 \subseteq X$ such that $K_i \subseteq V_i$ for $i = 1, 2$, as in any Hausdorff space. Hence there are open sets $W_1, W_2 \subseteq X$ such that $K_i \subseteq W_i$ for $i = 1, 2$ and $\overline{W_1}, \overline{W_2}$ are disjoint compact subsets of $X$.

### 37 $\sigma$-Compactness

Let $X$ be a topological space. A set $E \subseteq X$ is said to be $\sigma$-compact if there is a sequence of compact subsets $K_1, K_2, K_3, \ldots$ of $X$ such that $E = \bigcup_{i=1}^{\infty} K_i$. If $K \subseteq X$ is compact, then $K$ is automatically $\sigma$-compact, using $K_i = K$ for every $i \geq 1$. It is easy to see that $\sigma$-compact sets have the Lindelöf property, and that the union of a sequence of $\sigma$-compact sets is $\sigma$-compact as well. If $X$ is $\sigma$-compact, $X = \bigcup_{i=1}^{\infty} K_i$ for some compact sets $K_i$, and $E \subseteq X$ is closed, then $E$ is $\sigma$-compact, because $E \cap K_i$ is compact for each $i$ and $E = \bigcup_{i=1}^{\infty} (E \cap K_i)$. Similarly, if $X$ is $\sigma$-compact and $E$ is the union of a sequence of closed sets, then $E$ is $\sigma$-compact. If the topology on $X$ is determined by a metric, $X$ is $\sigma$-compact, and $U \subseteq X$ is an open set, then it follows that $U$ is $\sigma$-compact, since every open set in a metric space can be expressed as the union of a sequence of closed sets.
Suppose now that \( X \) is a locally compact topological space which is also \( \sigma \)-compact, so that
\[ X = \bigcup_{i=1}^{\infty} K_i \]
for some sequence of compact sets \( K_i, i \in \mathbb{Z}_+ \).
Local compactness implies that for each \( i \geq 1 \) there is an open set \( U_i \subseteq X \) and a compact set \( K_i \subseteq X \) such that \( K_i \subseteq U_i \subseteq \bar{K}_i \). Put \( V_i = \bigcup_{l=1}^{i} U_i \) and \( \bar{K}_l = \bigcup_{i=l}^{\infty} K_i \) for every \( l \geq 1 \), so that the \( V_i \)'s are open subsets of \( X \), the \( \bar{K}_l \)'s are compact subsets of \( X \), \( V_i \subseteq V_{i+1} \), \( \bar{K}_l \subseteq \bar{K}_{l+1} \), and \( V_l \subseteq \bar{K}_l \) for every \( l \geq 1 \), and
\[ X = \bigcup_{i=1}^{\infty} V_i = \bigcup_{l=1}^{\infty} \bar{K}_l. \]
The \( V_i \)'s form an open covering of \( X \), and therefore every compact set in \( X \) is contained in \( V_n \) for some positive integer \( n \), depending on the compact set. If one assumes that \( X \) is Hausdorff too, then one can take \( K_i = U_i \) for each \( i \). In this case, one can make some additional adjustments to the construction to get a sequence of open subsets \( W_1, W_2, W_3, \ldots \) of \( X \) such that \( X = \bigcup_{j=1}^{\infty} W_j \) and \( W_j \) is a compact set contained in \( W_{j+1} \) for every \( j \geq 1 \). Note that a topological space is \( \sigma \)-compact when it is locally compact and has the Lindelöf property.

Of course, the real line \( \mathbb{R} \) is \( \sigma \)-compact with respect to the standard topology, since closed and bounded intervals are compact. It follows that \( \mathbb{R} \) has the Lindelöf property, which one could also get from the fact that the open intervals \( (a,b) \) with \( a < b \) being rational numbers is a countable base for the topology of \( \mathbb{R} \). If the set of irrational numbers were \( \sigma \)-compact, then it would be the union of a sequence of closed subsets of \( \mathbb{R} \) in particular. This would imply that the set of rational numbers is the intersection of a sequence of open subsets of \( \mathbb{R} \), contradicting the Baire category theorem. However, the set of irrational numbers has the Lindelöf property, and in fact every subset of the real line has the Lindelöf property, since there is a countable base for the topology of \( \mathbb{R} \).

### 38 Topological manifolds

Let \( n \) be a positive integer, and let \( \mathbb{R}^n \) be the space of \( n \)-tuples of real numbers, which is the same as the Cartesian product of \( n \) copies of the real line. This leads to a natural topology on \( \mathbb{R}^n \), which is the product topology corresponding to the standard topology on the real line. This topology is also the one determined by the standard Euclidean metric on \( \mathbb{R}^n \). Note that the product of \( n \) closed intervals in \( \mathbb{R} \) determines a compact subset of \( \mathbb{R}^n \), which implies that \( \mathbb{R}^n \) is locally compact and \( \sigma \)-compact. One can also get a countable base for the standard topology on \( \mathbb{R}^n \), using such a base for the standard topology on \( \mathbb{R} \), or the fact that \( \mathbb{R}^n \) is a separable metric space.

A topological space \( X \) is said to be \textit{locally Euclidean} with dimension \( n \) if for every \( p \in X \) there is an open set \( U \subseteq X \) such that \( p \in U \) and \( U \) is homeomorphic to an open set in \( \mathbb{R}^n \), with respect to the standard topology on \( \mathbb{R}^n \), as in the previous paragraph. In this case, one might also say that \( X \) is an \textit{n-dimensional topological manifold}, although one often asks that \( X \) be Hausdorff as a topological space too. Another condition that is frequently included is that
there be a base for the topology of $X$ with only finitely or countably many elements. If $X$ is locally Euclidean, then $X$ is locally compact, because $\mathbb{R}^n$ is locally compact. Remember that $X$ has the Lindelöf property when there is a base for the topology of $X$ with only finitely or countably many elements, and that a locally compact topological space with the Lindelöf property is $\sigma$-compact.

Of course, $\mathbb{R}^n$ is locally Euclidean with dimension $n$ with respect to the standard topology, as is any open subset of $\mathbb{R}^n$ with respect to the corresponding induced topology. It is well known that the unit sphere in $\mathbb{R}^{n+1}$ with respect to the standard Euclidean metric is locally Euclidean with dimension $n$ too.

### 38.1 Unions of bases

Let $X$ be a topological space, and suppose that $\{U_i\}_{i \in I}$ is a collection of open subsets of $X$ such that $X = \bigcup_{i \in I} U_i$. If for every $i \in I$ there is a collection $\mathcal{B}_i$ of open subsets of $U_i$ which form a base for the topology of $U_i$, then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is a base for the topology of $X$. For let $p \in X$ and an open set $V \subseteq X$ with $p \in V$ be given. By hypothesis, $p \in U_i$ for some $i \in I$, and hence there is an open set $W \in \mathcal{B}_i$ such that $p \in W$ and $W \subseteq U_i \cap V$, since $\mathcal{B}_i$ is a base for the topology of $U_i$. In particular, $W \in \mathcal{B}$, $p \in W$, and $W \subseteq V$, as desired.

Suppose now that $X$ is locally Euclidean with dimension $n$. If an open set $U \subseteq X$ is homeomorphic to an open set in $\mathbb{R}^n$, then there is a collection $\mathcal{B}_U$ of countably many open subsets of $U$ that forms a base for the topology of $R^n$, as mentioned earlier. If $X$ has the Lindelöf property, then $X$ can be covered by a family of finitely or countably many such open subsets $U$. This implies that there is a countable base for the topology of $X$, by the remarks in the previous paragraph.

### 39 $\sigma$-Compactness and normality

Let $X$ be a locally compact Hausdorff topological space which is $\sigma$-compact, and let $W_1, W_2, \ldots$ be a sequence of open subsets of $X$ such that $W_j$ is a compact set contained in $W_{j+1}$ for every $j \geq 1$ and $X = \bigcup_{j=1}^{\infty} W_j$. Suppose that $A, B$ are disjoint closed subsets of $X$, and put $A_j = A \cap W_j$ and $B_j = B \cap W_j$ for every positive integer $j$. Hence $A_j$, $B_j$ are disjoint compact subsets of $W_{j+1}$ with

$$A_j \subseteq A \subseteq X \setminus B \quad \text{and} \quad B_j \subseteq B \subseteq X \setminus A$$

for each $j$. Because $A_1$, $B_1$ are disjoint compact subsets of $X$ and $X$ is Hausdorff, there are disjoint open subsets $U_1$, $V_1$ of $X$ such that $A_1 \subseteq U_1$, $B_1 \subseteq V_1$. We may as well suppose that $U_1 \subseteq W_2 \cap (X \setminus B)$ and $V_1 \subseteq W_2 \cap (X \setminus A)$, by replacing $U_1$, $V_1$ with their intersections with $W_2 \cap (X \setminus B)$, $W_2 \cap (X \setminus A)$, respectively. By shrinking $U_1$, $V_1$ further, using the hypothesis that $X$ is a locally compact Hausdorff topological space, we may suppose that $\overline{U_1}, \overline{V_1}$ are disjoint compact subsets of $X$ which satisfy $\overline{U_1} \subseteq W_2 \cap (X \setminus B)$, $\overline{V_1} \subseteq W_2 \cap (X \setminus B)$. In general,
we would like to choose disjoint open subsets \( U_j, V_j \) of \( X \) recursively in such a way that \( A_j \subseteq U_j, B_j \subseteq V_j, U_{j-1} \subseteq U_j \) and \( V_{j-1} \subseteq V_j \) when \( j \geq 2 \),

\[
U_j \subseteq W_{j+1} \cap (X \setminus B), \quad V_j \subseteq W_{j+1} \cap (X \setminus A),
\]

and \( U_j, V_j \) are disjoint compact sets for every positive integer \( j \). If \( j \geq 2 \) and we have chosen \( U_{j-1}, V_{j-1} \) in this manner, then

\[
\tilde{A}_j = A_j \cup U_{j-1}, \quad \tilde{B}_j = B_j \cup V_{j-1}
\]

are disjoint compact subsets of \( X \) such that

\[
\tilde{A}_j \subseteq W_{j+1} \cap (X \setminus B), \quad \tilde{B}_j \subseteq W_{j+1} \cap (X \setminus A),
\]

which permits us to repeat the process. It follows that \( U = \bigcup_{j=1}^{\infty} U_j, V = \bigcup_{j=1}^{\infty} V_j \) are disjoint open subsets of \( X \) which contain \( A, B \), respectively. Therefore \( X \) is normal.

## 40 Separating points

Let \( X \) be a topological space, and let \( C(X) \) be the space of continuous real-valued functions on \( X \), as usual. We say that \( C(X) \) separates points in \( X \) if for every \( p, q \in X \) with \( p \neq q \) there is an \( f \in C(X) \) such that \( f(p) \neq f(q) \). If \( C(X) \) separates points in \( X \), then \( X \) has to be a Hausdorff topological space.

For suppose that \( p, q \in X \) and \( f(p) \neq f(q) \). If \( f(p) < f(q) \), then there is an \( r \in \mathbb{R} \) such that \( f(p) < r < f(q) \). In this case, \( U = \{ x \in X : f(x) < r \} \) and \( V = \{ x \in X : f(x) > r \} \) are disjoint open subsets of \( X \) containing \( p \) and \( q \), respectively. The case where \( f(p) > f(q) \) can be handled similarly, and one can refine the argument a bit to get that \( X \) is completely Hausdorff. If \( C(X) \) separates points on \( X \) and \( Y \subseteq X \), then it is easy to see that \( C(Y) \) separates points on \( Y \) with respect to the induced topology. If \( \tau_1, \tau_2 \) are topologies on \( X \) such that continuous real-valued functions on \( X \) with respect to \( \tau_1 \) separate points and \( \tau_1 \subseteq \tau_2 \), then continuous real-valued functions on \( X \) with respect to \( \tau_2 \) separate points as well.

### 40.1 Some examples

On the real line, the continuous function \( f(x) = x \) is sufficient to separate points. Similarly, if \( j \) and \( n \) are positive integers with \( j \leq n \), then the \( j \)th standard coordinate function on \( \mathbb{R}^n \) sends \( x = (x_1, \ldots, x_n) \) to \( x_j \). These are continuous real-valued functions on \( \mathbb{R}^n \) with respect to the standard topology on \( \mathbb{R}^n \), which separate points in \( \mathbb{R}^n \).

Let \( n \) be a positive integer again, and let \( X \) be a Hausdorff topological space which is locally Euclidean with dimension \( n \). Also let \( W \subseteq X \) be an open set which is homeomorphic to an open set \( W' \subseteq \mathbb{R}^n \), and let \( \phi' \) be a continuous real-valued function on \( \mathbb{R}^n \) for which the set of \( x \in \mathbb{R}^n \) with \( \phi'(x) \neq 0 \) is
contained in a compact set $K'$ with $K' \subseteq W'$. We can restrict $\phi'$ to $W'$ and use the homeomorphism between $W$ and $W'$ to get a continuous real-valued function $\phi$ on $W$ corresponding to $\phi'$ and a compact set $K \subseteq W$ corresponding to $K'$ such that $K$ contains every $x \in W$ with $\phi(x) \neq 0$. If we put $\phi(x) = 0$ when $x \in X \setminus W$, then one can check that $\phi$ is a continuous real-valued function on $X$. In particular, one can use functions like these to separate points in $X$.

If $(M, d(x, y))$ is a metric space, then one can use the triangle inequality to check that $f_p(x) = d(x, p)$ is a continuous function on $M$ for every $p \in M$. It is easy to see that these functions separate points in $M$.

If $X$ is any set, then the indicator function $1_A$ associated to a subset $A$ of $X$ is defined on $X$ by putting $1_A(x)$ equal to 1 when $x \in X$, and 0 when $x \in X \setminus A$. If $X$ is a topological space, then it is easy to see that the indicator function $1_A$ associated to $A \subseteq X$ is continuous if and only if $A$ is both open and closed in $X$. A topological space $X$ is said to be totally separated if for every pair of points $p, q \in X$ with $p \neq q$, there is a set $A \subseteq X$ which is both open and closed such that $p \in A$ and $q \in X \setminus A$, in which case the corresponding indicator functions separate points in $X$.

41 Urysohn’s lemma

Let $X$ be a topological space. If $f$ is a continuous real-valued function on $X$ and $a, c$ are real numbers such that $a < c$, then the sets of $x \in X$ such that $f(x) \leq a$, $f(x) \geq c$ are disjoint closed subsets of $X$. If moreover $b \in \mathbb{R}$ and $a < b < c$, then the sets of $x \in X$ such that $f(x) < b$, $f(x) > b$ are disjoint open subsets of $X$ which contain the previous closed sets, respectively.

Conversely, suppose that $X$ is normal, and that $E_0$, $E_1$ are disjoint closed subsets of $X$. Let $\{r_i\}_{i=1}^\infty$ be an enumeration of the rational numbers $r$ such that $0 < r < 1$, i.e., $r_i \in \mathbb{Q} \cap (0, 1)$ for every $i \geq 1$, and every element of $\mathbb{Q} \cap (0, 1)$ is equal to $r_i$ for exactly one $i$. Because $X$ is normal, there is an open set $U_r \subseteq X$ such that $E_0 \subseteq U_r$, and $\overline{U_r} \subseteq X \setminus E_1$. Similarly, there is an open set $U_r \subseteq X$ such that $E_0 \subseteq U_r$, $\overline{U_r} \subseteq X \setminus E_1$, and $\overline{U_r} \subseteq U_r$ if $r_1 < r_2$, $U_r \subseteq U_{r_2}$ if $r_1 > r_2$. Continuing in this way, we get an open set $U_r \subseteq X$ for every $r \in \mathbb{Q} \cap (0, 1)$ such that $E_0 \subseteq U_r$, $\overline{U_r} \subseteq X \setminus E_1$, and $\overline{U_r} \subseteq U_r$ when $r < t$. Consider the real-valued function defined on $X$ by putting $f(x)$ equal to the infimum of $r \in \mathbb{Q} \cap (0, 1)$ such that $x \in U_r$ when there is such an $r$, and $f(x) = 1$ otherwise. Equivalently, $f(x)$ is equal to the supremum of $r \in \mathbb{Q} \cap (0, 1)$ such that $x \in X \setminus \overline{U_r}$ when there is such an $r$, and $f(x) = 0$ otherwise. By construction, $f(x) = 0$ when $x \in E_0$, $f(x) = 1$ when $x \in E_1$, and $0 \leq f(x) \leq 1$ for every $x \in X$. One can check that $f$ is a continuous function on $X$, using the fact that $U_r$, $X \setminus \overline{U_r}$ are open subsets of $X$ for every $r \in \mathbb{Q} \cap (0, 1)$. If $X$ is a locally compact Hausdorff topological space, $K \subseteq X$ is compact, $W \subseteq X$ is an open set, and $K \subseteq W$, then one can use analogous arguments to show that there is a continuous real-valued function $h$ on $X$ such that $h(x) = 1$ when $x \in K$, $h(x) = 0$ when $x \in X \setminus W$, and $0 \leq h(x) \leq 1$ for every $x \in X$. 

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41.1 Complete regularity

A topological space $X$ is said to be *completely regular* if it satisfies the first separation condition, and if for every point $p \in X$ and closed set $E \subseteq X$ with $p \notin E$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(p) \neq 0$ and $f(x) = 0$ for every $x \in E$. Of course, one may as well ask that $f(p) = 1$, by multiplying $f$ by a nonzero real number, if necessary. One can also ask that $0 \leq f(x) \leq 1$ for every $x \in X$, by taking the maximum of $f(x)$ and 0 and then the minimum of the result and 1. Urysohn’s lemma implies that normal topological spaces are completely regular, and it is easy to see that completely regular spaces are regular. Thus completely regular spaces are said to satisfy separation condition number three-and-a-half, and are also known as *Tychonoff spaces*.

It is easy to check that subspaces of completely regular spaces are completely regular as well. In particular, subspaces of normal spaces are completely regular. Locally compact Hausdorff topological spaces are completely regular, by the variant of Urysohn’s lemma for them mentioned previously. If $X$ is completely regular, $K \subseteq X$ is compact, $E \subseteq X$ is closed, and $E \cap K = \emptyset$, then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for every $x \in E$ and $f(y) = 1$ for every $y \in K$. Indeed, one can use compactness of $K$ and complete regularity to cover $K$ by finitely many open sets that can be separated from $E$ by suitable continuous functions, and then combine these functions to separate $K$ from $E$. If $X$ is a topological space where continuous real-valued functions separate points, then one can use similar arguments to show that $X$ satisfies the analogue of complete regularity in which $E$ is also required to be compact. One can then repeat the process to separate a pair of disjoint compact subsets of $X$ by a continuous real-valued function on $X$.

41.2 $\mathbb{R} \setminus \{0\}$

Consider the real line with the standard topology. The standard topology on $\mathbb{R}$ is determined by the standard metric, and in particular $\mathbb{R}$ is a normal topological space. The set $\mathbb{R} \setminus \{0\}$ of nonzero real numbers is a topological space too, with the topology induced by the standard topology on the real line. This topology is also determined by the restriction of the standard metric to $\mathbb{R} \setminus \{0\}$, and thus $\mathbb{R} \setminus \{0\}$ is normal. As subsets of the real line, the sets of positive and negative real numbers are open sets. As subsets of the set of nonzero real numbers, these sets are both open and closed. The function equal to 0 on the negative real numbers and equal to 1 on the positive real numbers is a continuous function on the set of nonzero real numbers. Of course, this function is not the restriction to the nonzero real numbers of a continuous function on the real line.

42 Countable bases

Let $X$ be a topological space, and suppose that $\mathcal{B}$ is a base for the topology of $X$ with only finitely or countably many elements. This is a very nice property for a topological space to have, as we have seen. Suppose that $X$ is also regular, and
let $U$ be an open set in $X$. If $p \in U$, then there is an open set $V(p)$ in $X$ such that $p \in V(p)$ and $V(p) \subseteq U$, by regularity. Thus $U = \bigcup \{V(p) : p \in U\}$, and it follows that there is a set $A \subseteq U$ with only finitely or countably many elements such that $U = \bigcup \{V(p) : p \in A\}$, since $B$ has only finitely or countably many elements. This implies that $U = \bigcup \{V(p) : p \in A\}$, so that every open set in $X$ can be expressed as the union of only finitely or countably many closed sets under these conditions. Equivalently, every closed set in $X$ can be expressed as the intersection of only finitely or countably many open sets. If $X$ is a locally compact Hausdorff space, then the same argument shows that every open set in $X$ can be expressed as the union of only finitely or countably many closed sets under these conditions. Equivalently, every closed set in $X$ can be expressed as the intersection of only finitely or countably many open sets. If $X$ is a locally compact Hausdorff space, then the same argument shows that every open set in $X$ is $\sigma$-compact.

A famous theorem of Urysohn states that if $X$ is normal and has a countable base for its topology, then there is a metric on $X$ that determines the same topology on $X$. This was extended to regular spaces with countable bases by Tychonoff.

### 43 One-point compactification

If $X$ is a locally compact Hausdorff topological space which is not compact, then the one-point compactification of $X$ is the topological space $X^*$ defined as follows. As a set, $X^*$ consists of the elements of $X$ together with an additional point $p^*$, the point at infinity. The open subsets of $X$ are also considered to be open subsets of $X^*$, and conversely they are the only open subsets of $X^*$ contained in $X$, i.e., which do not contain $p^*$. If $V \subseteq X^*$ and $p^* \in V$, then $V$ is considered to be an open set in $X^*$ if and only if $V \cap X$ is an open set in $X$ and there is a compact set $K \subseteq X$ such that $X \setminus K \subseteq V \cap X$. It is easy to see that this defines a topology on $X^*$. Specifically, the intersection of finitely many open subsets of $X^*$ that contain $p^*$ is an open set in $X^*$ because the union of finitely many compact subsets of $X$ is a compact set in $X$. By construction, $X$ is a dense open set in $X^*$. One can check that $X^*$ is Hausdorff using the hypothesis that $X$ is locally compact. If $\{V_i\}_{i \in I}$ is any open covering of $X^*$, then there is an $i^* \in I$ such that $p^* \in V_{i^*}$. And thus there is a compact set $K \subseteq X$ such that $X \setminus K \subseteq V_{i^*} \cap X$. Since $\{V_i \cap X\}_{i \in I}$ is an open covering of $X$, and hence of $K$, there is a collection $i_1, \ldots, i_n$ of finitely many elements of $I$ such that $K \subseteq \bigcup_{i=1}^n (V_{i} \cap X)$. It follows that $X^* \subseteq V_{i^*} \cup \bigcup_{i=1}^n V_i$, and therefore $X^*$ is compact.

#### 43.1 Supports of continuous functions

Let $X$ be a topological space, and let $C(X)$ be the space of continuous real-valued functions on $X$, as usual. The support of $f \in C(X)$ is denoted $\text{supp } f$ and defined to be the closure of the set of $x \in X$ such that $f(x) \neq 0$. If $f_1, f_2 \in C(X)$, then

$$\text{supp } f_1 + f_2 \subseteq (\text{supp } f_1) \cup (\text{supp } f_2)$$
Let us specialize to the situation where $X$ is a locally compact Hausdorff topological space. The space of continuous real-valued functions on $X$ with compact support is denoted $C_{\text{com}}(X)$. If $f_1, f_1 \in C_{\text{com}}(X)$, then $f_1 + f_2 \in C_{\text{com}}(X)$, by the previous observation. Similarly, if $f_1, f_2 \in C(X)$ and at least one of $f_1, f_2$ has compact support in $X$, then the product $f_1 f_2$ also has compact support. Let us say that $f \in C(X)$ vanishes at infinity if for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ when $x \in X \setminus K$. A continuous function with compact support automatically vanishes at infinity. The space of continuous real-valued functions on $X$ which vanish at infinity is denoted $C_0(X)$. If $f_1, f_2 \in C_0(X)$, then it is not too difficult to show that $f_1 + f_2 \in C_0(X)$. If $f_1 \in C_0(X)$, and if $f_2 \in C(X)$ is bounded, which is to say that there is an $A \geq 0$ such that $|f_2(x)| \leq A$ for every $x \in X$, then $f_1 f_2 \in C_0(X)$. Of course $C_{\text{com}}(X) = C_0(X) = C(X)$ when $X$ is compact. Suppose that $X$ is not compact, and let $X^*$ be the one-point compactification of $X$. If $f \in C_0(X)$, then $f$ extends to a continuous function on $X^*$ which is equal to 0 at the point $p^*$ at infinity, and conversely the restriction of any such function on $X^*$ to $X$ lies in $C_0(X)$. In the same way, continuous functions on $X$ with compact support correspond to continuous functions on $X^*$ which are equal to 0 on a neighborhood of $p^*$.

## 44 Connectedness

### 44.1 Connected topological spaces

A topological space $X$ is not connected if there are nonempty disjoint open subsets $U_1, U_2$ of $X$ such that

$$U_1 \cup U_2 = X.$$ 

Equivalently, $X$ is not connected if there are nonempty disjoint closed subsets $E_1, E_2$ of $X$ such that

$$E_1 \cup E_1 = X.$$ 

This is also the same as having a set $E \subseteq X$ which is both open and closed such that $E \neq \emptyset, X$. Otherwise, if $X$ does not have any of these equivalent properties, then $X$ is connected.

For example, if $X$ is equipped with the discrete topology and $X$ has at least two elements, then $X$ is not connected. If $X$ is equipped with the indiscrete topology, then $X$ is connected. If $X$ is an infinite set with the topology in which $U \subseteq X$ is an open set when $U = \emptyset$ or $X \setminus U$ has only finite many elements, then $X$ is connected, because

$$U_1 \cap U_2 \neq \emptyset$$

for any nonempty open subsets $U_1, U_2$ of $X$.

The real line with the standard topology is connected. For suppose that $E_1, E_2$ are nonempty disjoint closed subsets of $\mathbb{R}$ whose union is $\mathbb{R}$, and let $x_1, x_2$
be elements of $E_1$, $E_2$, respectively. We may as well suppose that $x_1 < x_2$, since otherwise we can relabel the indices. One can check that
\[
\sup(E_1 \cap [x_1, x_2]) \in E_1 \cap E_2,
\]
contradicting the disjointness of $E_1$ and $E_2$.

44.2 Connected sets

Let $X$ and $Y$ be topological spaces, and suppose that $f : X \to Y$ is a continuous mapping such that $f(X) = Y$. If $Y$ is not connected, then there are disjoint nonempty open subsets $V_1, V_2$ of $Y$ such that $V_1 \cup V_2 = Y$. This implies that $X$ is not connected, because $f^{-1}(V_1), f^{-1}(V_2)$ are disjoint nonempty open subsets of $X$ whose union is equal to $X$. This is the same as saying that the connectedness of $X$ implies the connectedness of $Y$ under these conditions.

A set $E$ in a topological space $X$ is said to be connected if it is connected as a topological space itself, using the topology induced from the one on $X$. A more direct characterization can be given as follows. A pair of subsets $A, B$ of $X$ are said to be separated if

\[ A \cap B = A \cap \overline{B} = \emptyset, \]

where $A$ and $B$ are the closures of $A$ and $B$ in $X$. One can check that $E \subseteq X$ is not connected if and only if $E$ is the union of a pair of nonempty separated subsets of $X$. This uses the fact that the closure of $A \subseteq E$ relative to $E$ is equal to the intersection of $E$ with the closure of $A$ relative to $X$, because a point $p \in E$ is adherent to $A$ as a subset of $E$ if and only if $p$ is adherent to $A$ as a subset of $X$. In particular, a pair of sets $A, B \subseteq E$ is separated relative to $E$ if and only if they are separated relative to $X$. It is easy to see that a pair of disjoint closed subsets of a topological space are separated, as well as a pair of disjoint open sets. Conversely, if a topological space is the union of a pair of separated sets, then the two sets are both open and closed.

Of course, one can also define connectedness of a subset $E$ of a topological space $X$ directly in terms of separated sets as in the previous paragraph, which is to say that $E \subseteq X$ is not connected if there are nonempty separated sets $A, B \subseteq X$ such that $A \cup B = E$. If $E \subseteq Y \subseteq X$, then one can check that $E$ is connected as a subset of $Y$, with the topology induced from the one on $X$, if and only if $E$ is connected as a subset of $X$. This is because $A, B \subseteq Y$ are separated relative to $Y$ if and only if $A, B$ are separated relative to $X$, since their closures relative to $Y$ are the same as the intersection of their closures relative to $X$ with $Y$, as before. In particular, $E$ is connected as a subset of $X$ if and only if $E$ is connected as a subset of itself, with the topology induced by the one on $X$.

If $X$ and $Y$ are topological spaces, $f : X \to Y$ is continuous, and $E$ is a connected subset of $X$, then $f(E)$ is connected in $Y$. This can be derived from the analogous statement with $E = X$ mentioned earlier, or shown directly using the characterization of connected subsets of topological spaces in terms of
separated sets. As an extension of the connectedness of the real line, one can show that a set $E \subseteq \mathbb{R}$ is connected with respect to the standard topology on $\mathbb{R}$ if and only if for every $x, y \in E$ with $x < y$, the interval $[x, y]$ is contained in $E$. The “only if” part of this statement is easy to check, while the “if” part is very similar to the connectedness of $\mathbb{R}$. Now let $a, b$ be real numbers such that $a < b$, and let $f$ be a continuous real-valued function on $[a, b]$. One can also think of $f$ as a continuous function on the real line, by putting $f(x) = f(a)$ when $x < a$, and $f(x) = f(b)$ when $x > b$. If $t$ is a real number such that $f(a) < t < f(b)$ or $f(b) < t < f(a)$, then the intermediate value theorem implies that there is an $x \in (a, b)$ such that $f(x) = t$. This follows from the connectedness of $f([a, b])$, using the connectedness of $[a, b]$ and the continuity of $f$.

44.3 Other properties

Let $X$ be a topological space. If $E_1, E_2$ are connected subsets of $X$ and $E_1 \cap E_2$ is not empty, then $E_1 \cup E_2$ is also connected. For suppose that there are nonempty separated sets $A, B$ of $X$ such that

$$A \cup B = E_1 \cup E_2.$$ 

It is easy to see that

$$A_1 = A \cap E_1, \quad B_1 = B \cap E_1$$

and

$$A_2 = A \cap E_2, \quad B_2 = B \cap E_2$$

are pairs of separated sets in $X$. The remaining point is that $A_1, B_1 \neq \emptyset$ or $A_2, B_2 \neq \emptyset$.

Similarly, if $E$ is a connected set in $X$, then one can show that the closure $\overline{E}$ of $E$ in $X$ is connected. Otherwise, there are nonempty separated sets $A, B$ in $X$ such that $A \cup B = \overline{E}$. If $A_0 = A \cap E$ and $B_0 = B \cap E$, then $A_0$ and $B_0$ are separated sets in $X$ too. Again the remaining point is to check that $A_0, B_0 \neq \emptyset$ under these conditions.

If $E \subseteq X$ has the property that for every $p, q \in E$ there is a connected set $E(p, q) \subseteq E$ such that $p, q \in E(p, q)$, then $E$ is connected. As usual, if $E$ is not connected, then $E$ is the union of a pair of nonempty separated sets $A, B$ in $X$. Let $p, q$ be elements of $A, B$, respectively. It is easy to see that

$$A \cap E(p, q), \quad B \cap E(p, q)$$

are nonempty separated sets whose union is $E(p, q)$, contradicting the connectedness of $E(p, q)$.

44.4 Pathwise connectedness

A set $E$ in a topological space $X$ is said to be pathwise connected if for every $p, q \in E$ there are real numbers $a, b$ with $a \leq b$ and a continuous mapping
Here \([a, b]\) is the closed interval in the real line with endpoints \(a, b\), equipped with the topology induced by the standard topology on \(\mathbb{R}\). In particular, \([a, b]\) is connected, and so

\[ E(p, q) = f([a, b]) \]

is connected. It follows from a previous observation that \(E\) is connected when \(E\) is pathwise connected.

There are famous examples of closed sets in the plane that are connected but not pathwise connected. These examples also show that the closure of a pathwise-connected set may not be pathwise connected. Specifically,

\[ E = \{(x, y) \in \mathbb{R}^2 : x > 0, \ y = \sin(1/x)\} \]

is pathwise connected, while

\[ \overline{E} = E \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} \]

is not, because there is no continuous path in \(\overline{E}\) that connects any point in \(E\) to any point of the form \((0, y)\), \(-1 \leq y \leq 1\). Instead of the sine function, similar examples could be based on any nonconstant continuous periodic real-valued function on the real line.

Let \(a, b,\) and \(c\) be real numbers with \(a \leq b \leq c\), and let \(\phi, \psi\) be continuous mappings from the closed intervals \([a, b], [b, c]\) into a topological space \(X\), respectively. If \(\phi(b) = \psi(b)\), then it is easy to see that the function on \([a, c]\) equal to \(\phi\) on \([a, b]\) and to \(\psi\) on \([b, c]\) is also continuous as a mapping from \([a, c]\) into \(X\). In particular, the union of two pathwise-connected subsets of a topological space is pathwise connected when they have a point in common, just as for connected sets.

Remember that a set \(E \subseteq \mathbb{R}^n\) is said to be **convex** if for every \(x, y \in E\), the line segment in \(\mathbb{R}^n\) connecting \(x\) and \(y\) is contained in \(E\). Thus convex sets are always pathwise connected. Of course, a ball in \(\mathbb{R}^n\) with respect to the standard Euclidean metric is convex. Subsets of \(\mathbb{R}^n\) which are “rectangular boxes” in the sense that the are Cartesian products of intervals are also convex.

### 44.5 Connected components

Remember that a binary relation \(x \sim y\) on a set \(X\) is said to be an **equivalence relation** if it satisfies the following three conditions. First, this relation should be reflexive, in the sense that \(x \sim x\) for every \(x \in X\). Second, this relation should be symmetric, which is to say that \(x \sim y\) if and only if \(y \sim x\) for every \(x, y \in X\). Third, this relation should be transitive, so that \(x \sim y\) and \(y \sim z\) imply that \(x \sim z\) for every \(x, y, z \in X\). If \(x \sim y\) is an equivalence relation on \(X\) and \(x \in X\), then put

\[ E(x) = \{w \in X : w \sim x\}. \]

It is easy to see that \(E(x) = E(y)\) when \(x \sim y\), and that

\[ E(x) \cap E(y) = \emptyset \]
when \( x \neq y \). Of course, \( x \in E(x) \) for each \( x \in X \), by reflexivity, so that

\[
\bigcup_{x \in X} E(x) = X.
\]

Subsets of \( X \) of the form \( E(x) \) for some \( x \in X \) are known as equivalence classes associated to the equivalence relation \( x \sim y \). A collection of pairwise-disjoint subsets of \( X \) whose union is equal to \( X \) is called a partition of \( X \). Thus the collection of equivalence classes corresponding to an equivalence relation on \( X \) is a partition of \( X \), and conversely it is easy to see that any partition of \( X \) determines an equivalence relation for which the subsets of \( X \) in the partition are the equivalence classes.

Now let \( X \) be a topological space, and consider the binary relation defined on \( X \) by saying that \( x \sim_c y \) when there is a connected set \( E \subseteq X \) such that \( x, y \in E \). One can check that this is an equivalence relation on \( X \), using properties of connected sets discussed earlier. The equivalence classes in \( X \) corresponding to this equivalence relation are connected subsets of \( X \), known as the connected components of \( X \). Equivalently, the connected component \( E(x) \) of \( X \) containing a point \( x \in X \) is equal to the union of all of the connected subsets of \( X \) that contain \( x \) as an element. Connected components of \( X \) are maximal connected subsets of \( X \), and they are automatically closed subsets of \( X \), because the closure of a connected set in \( X \) is also connected.

Similarly, another binary relation on \( X \) may be defined by saying that \( x \sim_p y \) when \( x, y \in X \) can be connected by a continuous path in \( X \). This is also an equivalence relation on \( X \), and the equivalence classes associated to this relation are known as pathwise-connected components of \( X \). It is easy to see that these are pathwise-connected subsets of \( X \), which are maximal with respect to this property. Every pathwise-connected component of \( X \) is contained in a connected component of \( X \), but the connected components of \( X \) may be larger than the pathwise-connected components. Each connected component of \( X \) is in fact the union of the pathwise-connected components that it contains.

### 44.6 Local connectedness

If \( X \) is a topological space, then there are three basic types of local connectedness conditions that one might consider. In the first condition, we ask that for each point \( p \in X \) there be an open set \( V \subseteq X \) such that \( p \in V \) and every element \( q \) of \( V \) is contained in a connected subset of \( X \) that also contains \( p \). This is equivalent to asking that the connected components of \( X \) be open sets. The second condition is stronger, and asks that for each \( p \in X \) and open set \( U \subseteq X \) with \( p \in U \) there be an open set \( V \) in \( X \) such that \( p \in V \subseteq U \), and each element \( q \) of \( V \) is contained in a connected subset of \( U \) that also contains \( p \). This is the same as saying that every open subset of \( X \) satisfies the first condition, as a topological space itself, with respect to the topology induced from the one on \( X \). The third condition is simpler, and asks that for each \( p \in X \) and open set \( U \subseteq X \) with \( p \in U \) there be a connected open set \( W \) in \( X \) such that \( p \in W \subseteq U \). This
obviously implies the second condition, and it is easy to see that the converse also holds, by taking $W$ to be the connected component of $U$ that contains $p$. Thus one normally defines local connectedness in terms of either the second or third conditions, which are equivalent.

### 44.7 Local pathwise connectedness

As before, there are three basic types of local pathwise connectedness conditions for a topological space $X$ that one might consider. The first is that for each point $p \in X$ there be an open set $V \subseteq X$ such that $p \in V$ and every element $q$ of $V$ can be connected to $p$ by a continuous path in $X$. This is equivalent to asking that the pathwise-connected components of $X$ be open sets. Note that this implies that the pathwise-connected components of $X$ are also closed sets, since the complement of any pathwise-connected component in $X$ is equal to the union of the all of the other pathwise-connected components, which are open sets in this case. This condition also implies that the pathwise-connected components of $X$ are the same as the connected components of $X$. More precisely, if $E$ is a connected component of $X$, then $E$ is automatically the union of the pathwise-connected components of $X$ that it contains. If $E$ were to contain more than one pathwise-connected component, then it would have to be disconnected, since the pathwise-connected components are pairwise disjoint and open in this situation.

The second local pathwise connectedness condition asks that for each $p \in X$ and open set $U \subseteq X$ there be an open set $V$ in $X$ such that $p \in V \subseteq U$, and every element of $V$ can be connected to $p$ by a continuous path in $U$. This is the same as saying that every open set $U$ in $X$ satisfies the first condition described in the previous paragraph, as a topological space itself, with the topology induced by the one on $X$. The third condition asks that for each $p \in X$ and open set $U \subseteq X$ that contains $p$ there be a pathwise-connected open set $W$ in $X$ such that $p \in W \subseteq U$. This is equivalent to the second condition, since the third condition obviously implies the second condition, and one can get the converse by taking $W$ to be the pathwise-connected component of $U$ that contains $p$.

Thus one normally says that $X$ is locally pathwise connected if it satisfies either of the second or third conditions, described in the previous paragraph. If $X$ is locally pathwise connected, then every open set $U \subseteq X$ is too, as before. This implies that the pathwise-connected components of $U$ are the same as the connected components of $U$, as in the discussion of the first condition. In particular, if $U$ is connected, then it follows that $U$ is pathwise connected in this situation. Note that $\mathbb{R}^n$ is locally pathwise connected with respect to the standard topology for each positive integer $n$, and hence connected open subsets of $\mathbb{R}^n$ are also pathwise connected.
45 A little set theory

45.1 Mappings and their properties

Let $A$, $B$ be sets, and let $f$ be a function on $A$ with values in $B$, also known as a mapping from $A$ into $B$. We say that $f$ is one-to-one or injective if $f(a) = f(a')$ implies that $a = a'$ for every $a, a' \in A$. We say that $f$ maps $A$ onto $B$, or that $f$ is a surjection, if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$. If $f$ is a one-to-one mapping of $A$ onto $B$, then we say that $f$ is a bijection, or that $f$ is a one-to-one correspondence between $A$ and $B$. In general, $f(A)$ denotes the set of all $b \in B$ for which there is an $a \in A$ such that $f(a) = b$.

Suppose that $A$, $B$, $C$ are sets and $f : A \to B$ and $g : B \to C$ are mappings between them. In this case, the composition $g \circ f$ of $f$ and $g$ is the mapping from $A$ to $C$ defined by

$$ (g \circ f)(a) = g(f(a)) $$

for every $a \in A$. It is easy to see that the composition of two injections is an injection, and that the composition of two surjections is a surjection. Hence the composition of two bijections is a bijection as well. If $f : A \to B$ is a bijection, then the inverse mapping $f^{-1} : B \to A$ is defined by $f^{-1}(b) = a$ when $f(a) = b$. The inverse mapping is a bijection whose inverse is $f$. By construction, $f^{-1} \circ f$ is the identity mapping on $A$, and $f \circ f^{-1}$ is the identity mapping on $B$. Conversely, if there is a mapping $f^{-1} : B \to A$ such that $f^{-1} \circ f$ is the identity mapping on $A$ and $f \circ f^{-1}$ is the identity mapping on $B$, then $f$ is a bijection, and $f^{-1}$ is the same as the inverse mapping just described. If $f : A \to B$ and $g : B \to C$ are both bijections, then it is easy to see that

$$ (g \circ f)^{-1} = f^{-1} \circ g^{-1}. $$

45.2 One-to-one correspondences

Let $A$, $B$ be sets. If there is a one-to-one correspondence from $A$ onto $B$, then we may express this formally by

$$ \#A = \#B. $$

Of course, $\#A = \#A$ automatically, because the identity mapping on $A$ is a one-to-one correspondence of $A$ to itself. Also, $\#A = \#B$ is equivalent to $\#B = \#A$, because the inverse of a bijection is a bijection. Similarly, $\#A = \#B$ and $\#B = \#C$ imply $\#A = \#C$ for any sets $A$, $B$, and $C$, since the composition of two bijections is a bijection. Note that $A$ is a finite set with exactly $n$ elements for some positive integer $n$ if and only if $\#A = \#\{1, \ldots, n\}$. A set $A$ is countably infinite if and only if $\#A = \#\mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of positive integers.

If $B$ is the set of infinite binary sequences and $[0, 1]$ is the unit interval in the real line, then $\#B = \#[0, 1]$. To see this, let $f$ be the standard mapping from $B$ onto $[0, 1]$, which sends a binary sequence $b = \{b_i\}_{i=1}^\infty$ to the real number $f(b) = \sum_{i=1}^\infty b_i 2^{-i}$. For each positive integer $n$, let $A_n$ be the finite set of $b \in B$ such that $b_i = 0$ for every $i \geq n$ or $b_i = 1$ for every $i \geq n$. If $A = \bigcup_{n=1}^\infty A_n$, then
A is countably infinite. Let $E$ be the set of rational numbers $r \in [0, 1]$ such that $r$ is an integer multiple of $2^{-l}$ for some nonnegative integer $l$. The restriction of $f$ to $B \setminus A$ is a one-to-one mapping onto $[0, 1] \setminus E$. The restriction of $f$ to $A$ maps onto $E$ but is not one-to-one. However, $A$ and $E$ are both countably infinite, and so there is a one-to-one correspondence between them. Combining this with the restriction of $f$ to $B \setminus A$ leads to a one-to-one correspondence from $B$ onto $[0, 1]$, as desired.

45.3 One-to-one mappings

If there is a one-to-one mapping from a set $A$ into a set $B$, then this may be expressed formally by $\#A \leq \#B$. Thus $\#A = \#B$ implies $\#A \leq \#B$ and $\#B \leq \#A$. If $A$, $B$, $C$ are sets such that $\#A \leq \#B$ and $\#B \leq \#C$, then $\#A \leq \#C$, since the composition of two injections is also an injection.

Suppose that $A$, $B$ are nonempty sets, and that $f$ is a one-to-one mapping from $A$ into $B$. Fix $\alpha \in A$, and let $g : B \to A$ be the mapping defined by $g(b) = a$ when $a \in A$ and $b = f(a)$ and $g(b) = \alpha$ when $b \in B \setminus f(A)$. Clearly $g$ is a surjection. Hence $\#A \leq \#B$ implies that there is a mapping from $B$ onto $A$. Conversely, suppose that there is a mapping $g$ from $B$ onto $A$. Assuming the axiom of choice, there is a mapping $f : A \to B$ such that $g \circ f$ is the identity mapping on $A$. Therefore $f$ is one-to-one.

If $A$, $B$ are sets such that $\#A \leq \#B$ and $\#B \leq \#A$, then $\#A = \#B$. Without loss of generality, we may suppose that $B \subseteq A$, since $\#B \leq \#A$ means that there is a one-to-one correspondence between $B$ and a subset of $A$. In this case, $\#A \leq \#B$ implies that there is an injective mapping $f : A \to A$ such that $f(A) \subseteq B$. Let $A_1, A_2, \ldots$ be the sequence of subsets of $A$ defined by $A_1 = f(A)$ and $A_{l+1} = f(A_l)$ for $l \geq 1$. One can use induction to show that $A_{l+1} \subseteq A_l$ for each $l$. If $C = \bigcap_{l=1}^{\infty} A_l$, then it is easy to check also that $f(C) = C$. Because $C \subseteq B$, it suffices to show that there is a one-to-one correspondence from $A \setminus C$ onto $B \setminus C$. Put $E_1 = A \setminus f(A_1)$ and $E_l = A_l \setminus A_{l+1}$ when $l \geq 1$, and observe that $f(E_l) = E_{l-1}$ for each $l$. By construction, the $E_l$'s are pairwise disjoint, and $\bigcup_{l=1}^{\infty} E_l = A \setminus C$. For each $x \in E_1$, let $O(x)$ be the orbit of $x$ with respect to $f$, which is to say the set consisting of $x$, $f(x)$, $f(f(x))$, etc. These orbits are pairwise disjoint, and their union is $A \setminus C$. We would like to show that there is a one-to-one mapping from each orbit $O(x)$ onto $O(x) \cap B$. All but possibly the initial element $x$ of $O(x)$ is automatically contained in $B$, since $f(A) \subseteq B$. It follows that either the identity mapping or a simple shift is a one-to-one mapping of $O(x)$ onto $O(x) \cap B$, as desired.

45.4 Some basic properties

Let $A$, $B$, $C$ be sets, with $B \subseteq A$ and $A \cap C = \emptyset$. If $B$ is countably infinite and $C$ has only finitely or countably many elements, then $B \cup C$ is countably infinite. One can use this to show that $\#(A \cup C) = \#A$. For example, one can
apply this to check that

\[ \#(0, 1) = \#[0, 1) = \#(0, 1] = \#[0, 1). \]

Also, \( f(x) = 1/2 + x/(1 + 2|x|) \) is a one-to-one mapping from the real line \( \mathbb{R} \) onto the open interval \((0, 1)\), and hence \( \#\mathbb{R} = \#(0, 1) \). Remember that the set \( \mathbb{Q} \) of rational numbers is countable. As another example, one can check that \( \mathbb{R} \setminus \mathbb{Q} \) is finite.

In general, if \( A, C \) are sets, \( A \) is infinite, and \( \#C \leq \#A \), then there is an argument based on the axiom of choice which implies that \( \#(A \cup C) = \#A \).

Another argument based on the axiom of choice implies that \( \#A \leq \#B \) or \( \#B \leq \#A \) for any pair of sets \( A, B \).

### 45.5 Bases of topological spaces

Let \((X, \tau)\) be a topological space which has a countable base for its topology. Thus there is a sequence \( U_1, U_2, \ldots \) of open subsets of \( X \) such that every open set in \( X \) can be expressed as a union of some collection of the \( U_j \)'s. Of course, the union of any collection of the \( U_j \)'s is an open set in \( X \), since each \( U_j \) is open. Let \( B \) be the set of all binary sequences, and consider the mapping \( \phi : B \to \tau \) which sends a binary sequence \( b = (b_j)_{j=1}^\infty \) to the open set in \( X \) that is the union of the \( U_j \)'s such that \( b_j = 1 \). If \( b_j = 0 \) for each \( j \), then we interpret \( \phi(b) \) as being the empty set. Because the \( U_j \)'s form a base for the topology of \( X \), \( \phi \) maps \( B \) onto the collection \( \tau \) of all open subsets of \( X \). Similarly, let \( \psi : \tau \to B \) be the mapping that sends an open set \( V \) in \( X \) to the binary sequence which is equal to 1 for each positive integer \( j \) such that \( U_j \subseteq V \) and equal to 0 otherwise. Because the \( U_j \)'s form a base for the topology of \( X \), each open set \( V \) in \( X \) is equal to the union of the \( U_j \)'s such that \( U_j \subseteq V \). Equivalently, \( \phi(\psi(V)) = V \) for each open set \( V \) in \( X \), which implies that \( \psi \) is a one-to-one mapping of \( \tau \) into \( B \). Consequently, \( \#\tau \leq \#B \).

Suppose now that \((X, \tau)\) is the real line with the standard topology. For each real number \( t \), the set \((-\infty, t)\) of \( r \in \mathbb{R} \) with \( r < t \) is an open set in \( \mathbb{R} \). This defines a one-to-one mapping of \( \mathbb{R} \) into \( \tau \). Hence \( \#\mathbb{R} \leq \#\tau \), and therefore \( \#\tau = \#\mathbb{R} \), since the open intervals in \( \mathbb{R} \) with rational endpoints form a countable base for the topology.

### 45.6 Exponentials

Let \( A, B \) be sets. The set of all functions on \( A \) with values in \( B \) is sometimes denoted \( B^A \). This is the same as the Cartesian product of a family of copies of \( B \), with one copy of \( B \) for each element of \( A \). This may also be denoted \( 2^A \) when \( B = \mathbb{R} \) is the set consisting of exactly the two elements 0 and 1. The set of all subsets of \( A \) may be identified with \( 2^A \), where a set \( E \subseteq A \) corresponds to the function on \( A \) which is equal to 1 at elements of \( E \) and to 0 at elements of \( A \setminus E \). If \( A \) and \( B \) are finite sets, then \( B^A \) is a finite set. In this case, the number of elements of \( B^A \) is the number of elements of \( B \) to the power which is the number of elements of \( A \).
45.7 Properties of exponentials

Let $A$, $B$, and $C$ be sets. Remember that the Cartesian product $A \times B$ consists of the ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. There is a natural one-to-one correspondence between the set $C^{A \times B}$ of mappings from $A \times B$ into $C$ and the set $(C^B)^A$ of mappings from $A$ into the set of mappings from $B$ into $C$. For if $f(a, b)$ is a function on $A \times B$ with values in $C$, then for each $a \in A$ we can view $f_a(b) = f(a, b)$ as a function on $B$ with values in $C$. Equivalently, $a \mapsto f_a$ is a mapping from $A$ into $C^B$. Thus we get a mapping from $C^{A \times B}$ into $(C^B)^A$. It is easy to see that this is a one-to-one correspondence, as desired. If $A$ and $B$ are countably infinite, then $A \times B$ is countably infinite, and

$$\#(C^B)^A = \# C^{A \times B} = \# C^{Z_+}.$$

By definition, the set $B$ of binary sequences is the same as $2^{Z_+}$. Consider the set $Z_+^Z$ of all sequences of positive integers. One can check that

$$\# B = \# 2^{Z_+} \leq \# Z_+^Z \leq \# B^{Z_+},$$

since $2 \leq \# Z_+ \leq \# B$. However, the remarks in the previous paragraph imply that $\# B^{Z_+} = \# (2^{Z_+})^Z = \# 2^{Z_+ \times Z_+} = \# 2^{Z_+} = \# B$. Therefore $\# Z_+^Z = \# B$.

45.8 Countable dense sets

Suppose that $X$ is a Hausdorff topological space which satisfies the local countability condition at each point and contains a countable dense set $E$. Under these conditions, for each $x \in X$ there is a sequence $\{x_i\}_{i=1}^\infty$ of elements of $E$ that converges to $x$. Because $X$ is Hausdorff, the limit of any convergent sequence of elements of $X$ is unique. Let $B$ be the set of all binary sequences, and let $C$ be the set of convergent sequences of elements of $E$. Thus $C \subseteq E^{Z_+}$, and hence $\# C \leq \# E^{Z_+} = \# B$. There is a natural mapping from $C$ into $X$, which sends a convergent sequence to its limit in $X$. By hypothesis, this is a mapping from $C$ onto $X$. Therefore $\# X \leq \# C \leq \# B$.

If $X$ is the real line with the standard topology, then $X$ satisfies the conditions described in the previous paragraph, and $\# X = \# B$.

45.9 Additional properties of exponentials

Let $A$, $B$, and $C$ be sets, with $A \cap B = \emptyset$. There is a natural one-to-one correspondence between the set $C^{A \cup B}$ of mappings from $A \cup B$ into $C$ and the Cartesian product $C^A \times C^B$ of the set $C^A$ of mappings from $A$ into $C$ with the set $C^B$ of mappings from $B$ into $C$. Basically, a mapping from $A \cup B$ into $C$ is the same as a combination of a mapping from $A$ into $C$ and a mapping from $B$ into $C$. If $A$ and $B$ are countable sets, then $A \cup B$ is countable, and it follows that $\# C^A = \# C^B = \# C^{A \cup B}$. Thus $\# C^A \times C^B = \# C^{Z_+}$, and hence $\# C^{Z_+} \times C^{Z_+} = \# C^{Z_+}$.

In particular, the set $B$ of all binary sequences is the same as $C^{Z_+}$ with $C = \{0, 1\}$. Therefore $\# B \times B = \# B$. 

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46 Some more set theory

46.1 Zorn’s lemma

Let \((A, \prec)\) be a partially-ordered set. If \(C \subseteq A\), then the restriction of the ordering \(\prec\) to \(C\) also defines a partial ordering on \(C\). If the restriction of \(\prec\) to \(C\) is a linear ordering on \(C\), then \(C\) is said to be a chain in \(A\). The term “chain” may also be used to refer to any linearly ordered set, not just in the context of subsets of a fixed partially-ordered set. An element \(b\) of \(A\) is said to be an upper bound for \(C \subseteq A\) if \(x \prec b\) for every \(x \in C\). An element \(a\) of \(A\) is said to be maximal in \(A\) if for every \(c \in A\) with \(a \prec c\), we have that \(a = c\). If \(a \in A\) is an upper bound for \(A\), then \(a\) is automatically maximal in \(A\). If \(A\) is linearly ordered, then every maximal element of \(A\) is also an upper bound for \(A\), but this is not normally the case for partially-ordered sets.

Suppose that \(A\) is a partially-ordered set, and that every chain \(C\) in \(A\) has an upper bound in \(A\). Under these conditions, Zorn’s lemma asserts that \(A\) has a maximal element.

If \(A\) has only finitely many elements, then every chain \(C\) in \(A\) has only finitely many elements, and hence a maximal element which is an upper bound for \(C\). In this case, Zorn’s lemma can be understood directly, as follows. Let \(x_1\) be an element of \(A\). If \(x_1\) is a maximal element of \(A\), then there is nothing to do, and we stop. Otherwise, if \(x_1\) is not a maximal element of \(A\), then there is an \(a_2 \in A\) such that \(a_1 \prec a_2\) and \(a_1 \neq a_2\). If \(a_2\) is maximal in \(A\), then we stop, and otherwise we can repeat the process. If \(A\) has only finitely many elements, then the process has to stop in finitely many steps, leading to a maximal element of \(A\).

Suppose now that \(A\) is countably infinite, so that there is a sequence \(\{x_j\}_{j=1}^\infty\) of elements of \(A\) in which every element of \(A\) occurs exactly once. If \(x_1\) is a maximal element of \(A\), then we stop. Otherwise, put \(j_1 = 1\), and let \(j_2\) be the smallest positive integer such that \(j_2 > 1\) and \(x_1 \prec x_{j_2}\). If \(x_{j_2}\) is maximal, then we stop, and otherwise we repeat the process. In general, if \(j_1 < j_2 < \cdots < j_n\) have been chosen, and \(x_{j_n}\) is maximal in \(A\), then we stop. Otherwise, we let \(j_{n+1}\) be the smallest positive integer such that \(j_{n+1} > j_n\) and \(x_{j_n} \prec x_{j_{n+1}}\). If the process stops after finitely many steps, then we get a maximal element of \(A\). Otherwise, we get a subsequence \(\{x_{j_l}\}_{l=1}^\infty\) of \(\{x_j\}_{j=1}^\infty\) such that \(x_{j_l} \prec x_{j_{l+1}}\) for each \(l\). Thus the set \(C\) consisting of the \(x_{j_l}\)’s is a chain in \(A\). Suppose for the sake of a contradiction that \(b \in A\) is an upper bound for \(C\). Let \(k\) be the unique positive integer such that \(b = x_k\). Note that \(x_{j_l} \prec x_k\) for every \(l \in \mathbb{Z}_+\), because \(b\) is an upper bound for \(C\). This implies that \(k = j_p\) for some \(p\), by construction. Hence \(b = x_k = x_{j_p} \prec x_{j_{p+1}},\) which implies in turn that \(b = x_{j_{p+1}}\), because \(x_{j_{p+1}} \prec b\). This is a contradiction, since \(j_p < j_{p+1}\), so that \(x_{j_p} \neq x_{j_{p+1}}\).
46.2 Hausdorff’s maximality principle

Let $(A, \prec)$ be a partially ordered set again. The Hausdorff maximality principle states that there is a maximal chain $C_1$ in $A$. More precisely, this means that $C_1$ is a chain in $A$, and if $C$ is a chain in $A$ such that $C_1 \subseteq C$, then $C_1 = C$.

It is easy to see that the Hausdorff maximality principle implies Zorn’s lemma, because an upper bound for a maximal chain in $A$ is a maximal element of $A$.

Conversely, one can show that Zorn’s lemma implies the Hausdorff maximality principle, by considering the collection $E$ of chains in $A$ as a partially ordered set with respect to inclusion. Thus a chain $L$ in $E$ is a collection of chains in $A$, with the property that if $C, C' \in L$, then either $C \subseteq C'$ or $C' \subseteq C$. In this case, one can check that the union $\bigcup_{C \in L} C$ of the elements of $L$ is also a chain in $A$, which is an upper bound for $L$ as an element of $E$. This shows that $E$ satisfies the hypothesis of Zorn’s lemma, and the conclusion of Zorn’s lemma applied to $E$ is exactly the same as the existence of a maximal chain in $A$.

If $A$ is a partially ordered set with only finitely many elements, then it is easy to get a maximal chain in $A$, by taking a chain in $A$ with a maximal number of elements. Suppose now that $A$ is countably infinite, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of $A$ in which every element of $A$ occurs exactly once. Put $j_1 = 1$, and let $j_2$ be the smallest positive integer greater than 1 such that $x_1 \prec x_{j_1}$ or $x_{j_1} \prec x_1$, if there is one. In general, if $1 = j_1 < j_2 < \cdots < j_n$ have been chosen such that $\{x_{j_1}, \ldots, x_{j_n}\}$ is a chain in $A$, then we let $j_{n+1}$ be the smallest positive integer such that $j_{n+1} > j_n$ and $\{x_{j_1}, \ldots, x_{j_n}, x_{j_{n+1}}\}$ is also a chain in $A$, if there is one. It is easy to see that this process leads to a maximal chain in $A$, whether it stops after finitely many steps, or continues indefinitely.

46.3 Choice functions

Let $I$ be a nonempty set, and suppose that for each $i \in I$, $E_i$ is a nonempty set. Consider the collection $P$ of ordered pairs $(A, f)$, where $A$ is a nonempty subset of $I$, and $f$ is a function from $A$ into $\bigcup_{i \in A} E_i$ such that $f(i) \in E_i$ for each $i \in A$. If $(A, f), (B, g) \in P$, then put $(A, f) \prec (B, g)$ when $A \subseteq B$ and $f(i) = g(i)$ for every $i \in A$, so that $f$ is equal to the restriction of $g$ to $B$. It is easy to see that this defines a partial ordering on $P$. We would like to use Zorn’s lemma or the Hausdorff maximality principle to show that there is an element $(A, f)$ of $P$ with $A = I$.

Suppose that $L$ is a chain in $P$, and consider

$$A_L = \bigcup\{A : (A, f) \in L\}.$$ 

If $(A, f), (B, g) \in L$, then $(A, f) \prec (B, g)$ or $(B, g) \prec (A, f)$, because $L$ is a chain in $P$. Hence $f(i) = g(i)$ for every $i \in A \cap B$. If $i \in A_L$, then $i \in A$ for some $(A, f) \in L$, and we put $f_L(i) = f(i)$. This does not depend on the choice of $(A, f) \in L$ such that $i \in A$, by the previous observation. Equivalently, the graph of $f_L$ as a subset of the Cartesian product of $A_L$ and $\bigcup_{i \in A_L} E_i$ is the union of the graphs of the functions $f$ with $(A, f) \in L$. At any rate, this defines
a function \( f_L \) on \( A_L \) with values in \( \bigcup_{i \in A_L} E_i \) such that \( f_i \in E_i \) for each \( i \in A_L \). Thus \( (A_L, f_L) \in \mathcal{P} \), and \( (A, f) \prec (A_L, f_L) \) for every \((A, f) \in \mathcal{L}\) by construction.

This shows that every chain \( \mathcal{L} \) in \( \mathcal{P} \) has a upper bound, so that Zorn's lemma implies that there is a maximal element \((A_0, f_0)\) in \( \mathcal{P} \). Alternatively, the Hausdorff maximality principle implies that there is a maximal chain \( \mathcal{L}_0 \) in \( \mathcal{P} \), and if \((A_0, f_0) = (A_{\mathcal{L}_0}, f_{\mathcal{L}_0})\) is as in the previous paragraph, then it is easy to see that \((A_0, f_0)\) is maximal in \( \mathcal{P} \). If \( A_0 \neq I \), then let \( i_1 \) be an element of \( I \setminus A_0 \), and put \( A_1 = A_0 \cup \{i_1\} \). Also put \( f_1(i) = f_0(i) \) for every \( i \in A_0 \), and let \( f_1(i_1) \) be an element of \( E_{i_1} \). This defines a function \( f_1 \) on \( A_1 \) with values in \( \bigcup_{i \in A_1} E_i \) such that \( f_1(i) \in E_i \) for each \( i \in A_1 \), so that \((A_1, f_1)\) is \( \mathcal{P} \). By construction, \((A_0, f_0) \prec (A_1, f_1)\) and \((A_0, f_0) \neq (A_1, f_1)\), contradicting maximality of \((A_0, f_0)\).

Thus \( A_0 = I \), as desired.

This shows that the axiom of choice follows from Zorn's lemma or the Hausdorff maximality principle. Conversely, there are well-known arguments by which Zorn's lemma or the Hausdorff maximality principle may be derived from the axiom of choice.

### 46.4 Comparing sets

Let \( A, B \) be sets, and let \( \mathcal{P} \) be the collection of ordered pairs \((A_1, f_1)\) such that \( A_1 \subseteq A \) and \( f_1 \) is a one-to-one mapping from \( A_1 \) into \( B \). If \((A_1, f_1), (A_2, f_2) \in \mathcal{P} \), then put \((A_1, f_1) \prec (A_2, f_2)\) when \( A_1 \subseteq A_2 \) and \( f_1 \) is equal to the restriction of \( f_2 \) to \( A_1 \). It is easy to see that this defines a partial ordering on \( \mathcal{P} \).

Let \( \mathcal{L} \) be a chain in \( \mathcal{P} \), and put

\[
A_L = \bigcup \{A_1 : (A_1, f_1) \in \mathcal{L}\}.
\]

If \((A_1, f_1), (A_2, f_2) \in \mathcal{L}\), then \((A_1, f_1) \prec (A_2, f_2)\) or \((A_2, f_2) \prec (A_1, f_1)\), and hence \( f_1(x) = f_2(x) \) for every \( x \in A_1 \cap A_2 \). Thus we can define \( f_L : A_L \to B \) by \( f_L(x) = f_1(x) \) when \((A_1, f_1) \in \mathcal{L}\) and \( x \in A_1 \), as in the previous situation.

If \( x, y \in A_L \), then there are \((A_1, f_1), (A_2, f_2) \in \mathcal{L}\) such that \( x \in A_1 \) and \( y \in A_2 \). Again \((A_1, f_1) \prec (A_2, f_2)\) or \((A_2, f_2) \prec (A_1, f_1)\), because \( \mathcal{L} \) is a chain in \( \mathcal{P} \). In the first case, \( x, y \in A_2 \) and \( f_L(x) = f_2(x) \), \( f_L(y) = f_2(y) \). If \( f_L(x) = f_L(y) \), then \( f_2(x) = f_2(y) \), and hence \( x = y \), since \( f_2 : A_2 \to B \) is one-to-one. In the second case, \( x, y \in A_1 \) and \( f_L(x) = f_1(x) \), \( f_L(y) = f_1(y) \), and the injectivity of \( f_1 : A_1 \to B \) implies that \( x = y \) when \( f_L(x) = f_L(y) \). This shows that \( f_L : A_L \to B \) is injective, so that \((A_L, f_L) \in \mathcal{P}\). Of course, we also have that \((A_1, f_1) \prec (A_L, f_L)\) for every \((A_1, f_1) \in \mathcal{L}\), by construction.

Thus every chain \( \mathcal{L} \) in \( \mathcal{P} \) has an upper bound, so that Zorn's lemma implies that there is a maximal element \((A_0, f_0)\) in \( \mathcal{P} \). Alternatively, the Hausdorff maximality principle implies that there is a maximal chain \( \mathcal{L}_0 \) in \( \mathcal{P} \), and if \((A_0, f_0) = (A_{\mathcal{L}_0}, f_{\mathcal{L}_0})\) is as in the previous paragraph, then it is easy to see that \((A_0, f_0)\) is maximal in \( \mathcal{P} \). If \( A_0 \neq A \) and \( f_0(A_0) \neq B \), then let \( a \) be an element of \( A \setminus A_0 \), and let \( b \) be an element of \( B \setminus f_0(A_0) \). Put \( A'_0 = A_0 \cup \{a\} \), and define \( f'_0 : A'_0 \to B \) by \( f'_0(x) = f_0(x) \) when \( x \in A_0 \) and \( f'_0(a) = b \). Thus \( f'_0 \) is a one-to-one mapping from \( A'_0 \) into \( B \), and hence \((A'_0, f'_0) \in \mathcal{P}\). This contradicts
the maximality of \((A_0, f_0)\), since \((A_0, f_0) \prec (A'_0, f'_0)\) and \((A_0, f_0) \neq (A'_0, f'_0)\) by construction.

It follows that either \(A_0 = A\) or \(f_0(A_0) = B\). In the first case, \(f_0\) is a one-to-one mapping from \(A\) into \(B\), so that \(#A \leq #B\). In the second case, the inverse of \(f_0\) defines a one-to-one mapping from \(B\) into \(A\), and hence \(#B \leq #A\).

### 46.5 Well-ordered sets

A linearly-ordered set \((A, \prec)\) is said to be well-ordered if for every nonempty subset \(B\) of \(A\) there is a \(b \in B\) such that \(b \prec c\) for each \(c \in B\). Of course, every nonempty finite subset of a linearly ordered set has a minimal element, so that linearly-ordered sets with only finitely many elements are well-ordered. The set \(\mathbb{Z}_+\) of positive integers with the standard ordering is also well-ordered.

The well-ordering principle states that any set admits a well-ordering. There are well-known arguments by which this may be derived from the axiom of choice. Conversely, if \(A\) is equipped with a well-ordering, then one can get a mapping from the set of nonempty subsets of \(A\) into \(A\) that associates to each non-empty set \(B \subseteq A\) an element of \(B\) by taking the minimal element of \(B\) with respect to the ordering. Thus the well-ordering principle implies the axiom of choice.

Suppose that \(I\) is a partially-ordered set, and that for each \(i \in I\), \(A_i\) is a partially-ordered set. Suppose also that \(A_i \cap A_j = \emptyset\) when \(i \neq j\), which can easily be arranged by replacing \(A_i\) with \(A_i \times \{i\}\) if necessary. One can define a partial ordering on \(A = \bigcup_{i \in I} A_i\) in the following way. If \(x, y \in A\) are elements of \(A_i\) for the same \(i \in I\), then one can use the ordering on \(A_i\) to order \(x\) and \(y\) in \(A\). Otherwise, if \(x \in A_i\) and \(y \in A_j\) for some \(i, j \in I\) with \(i \neq j\), then \(x\) precedes \(y\) in the ordering on \(A\) if and only if \(i\) precedes \(j\) in the ordering on \(I\). It is easy to see that this defines a partial ordering on \(A\), using the corresponding properties of the partial orderings on \(I\) and each \(A_i\). Similarly, this defines a linear ordering on \(A\) when \(I\) and each \(A_i\) are linearly-ordered. If \(I\) and each \(A_i\) are well-ordered, then \(A\) is well-ordered too. For if \(B\) is a nonempty subset of \(A\), then

\[
B_I = \{i \in I : A_i \cap B \neq \emptyset\}
\]

is a nonempty subset of \(I\), and hence has a minimal element \(i_0\). The minimal element of \(B\) in \(A\) is then the same as the minimal element of \(B \cap A_{i_0}\) as a subset of \(A_{i_0}\).

If \((E_1, \prec_1)\) and \((E_2, \prec_2)\) are partially-ordered sets, then the lexicographic ordering \(\prec\) on \(E = E_1 \times E_2\) is defined as follows. If \(x_1, y_1 \in E_1\) and \(x_2, y_2 \in E_2\), then

\[
(x_1, x_2) \prec (y_1, y_2)
\]

when either \(x_1 \prec_1 y_1\) and \(x_1 \neq y_1\), or \(x_1 = y_1\) and \(x_2 \prec_2 y_2\). It is well-known and easy to check that this defines a partial ordering on \(E\), which is a linear ordering when \(\prec_1, \prec_2\) are linear orderings on \(E_1, E_2\), respectively. Similarly, \(E\) is well-ordered by \(\prec\) when \(E_1, E_2\) are well-ordered by \(\prec_1, \prec_2\), respectively. This
can also be seen as a special case of the discussion in the previous paragraph, with \( I = E_1 \) and \( A_i = \{ i \} \times E_2 \) for each \( i \in E_1 \).

### 46.6 Comparing well-ordered sets

Let \((A, \prec)\) be a well-ordered set, and let \( \alpha \) be an order automorphism on \( A \), which is to say a one-to-one mapping from \( A \) onto itself such that \( \alpha(x) \prec \alpha(y) \) for every \( x, y \in A \) with \( x \prec y \). If \( \alpha \) is not the identity mapping on \( A \), then the set of \( x \in A \) such that \( \alpha(x) \neq x \) is nonempty, and hence this set has a minimal element \( x_0 \). However, it is not difficult to show that \( \alpha(x_0) = x_0 \) under these conditions, from which one may conclude that \( \alpha \) is the identity mapping on \( A \). This implies that an order isomorphism between two well-ordered sets is necessarily unique.

A subset \( I \) of \( A \) is said to be an (order) ideal in \( A \) if for each \( x \in A \) and \( y \in I \) with \( x \prec y \) we have that \( x \in I \). If \( I, I' \) are ideals in \( A \) and \( \alpha \) is an order isomorphism from \( I \) onto \( I' \), then one can check that \( \alpha \) is the identity mapping, and hence that \( I = I' \), by basically the same type of argument as before. Of course, \( A \) is an ideal in itself, and so we get in particular that an order isomorphism \( \alpha \) from \( A \) onto an ideal \( I' \) in \( A \) is the identity mapping, and that \( I' = A \) in this case. This would not work without the hypothesis that \( I' \) be an ideal in \( A \), since there are plenty of order-preserving one-to-one mappings from the set \( \mathbb{Z}_+ \) of positive integers into itself.

If \( a \) is any element of \( A \), then it is easy to see that

\[
I_a = \{ x \in A : x \prec a, x \neq a \}
\]

is an ideal in \( A \). Conversely, if \( I \) is an ideal in \( A \), \( I \neq A \), and \( a \) is the minimal element of \( A \setminus I \), then \( I = I_a \). It follows that if \( I, I' \) are any two ideals in \( A \), then either \( I \subseteq I' \) or \( I' \subseteq I \).

Suppose now that \( A \) and \( B \) are well-ordered sets. A well-known theorem states that either \( A \) is isomorphic to an ideal in \( B \), or that \( B \) is isomorphic to an ideal in \( A \). This includes the possibility that \( A \) is isomorphic to \( B \), and otherwise \( A \) or \( B \) is isomorphic to a proper ideal in the other set. Note that the ideal and the isomorphism in the conclusion of this theorem are unique, by the preceding discussion. The basic idea of the proof of the theorem is quite simple, since the minimal element of \( A \) should correspond to the minimal element of \( B \), and so on.

### 46.7 Well-ordered subsets

Let \((A, \prec)\) be a well-ordered set, let \( I \) be an ideal in \( A \), and let \( \phi \) be a one-to-one order-preserving mapping from \( I \) into \( A \). Note that \( \phi(I) \) is not asked to be an ideal in \( A \) here. Let us check that

\[
x \prec \phi(x)
\]

for every \( x \in I \) under these conditions. Otherwise, there is a minimal element \( y \) of \( I \) such that \( \phi(y) \prec y \) and \( \phi(y) \neq y \). Let \( I_y \) be the set of \( x \in A \) such that
$x \prec y$ and $x \neq y$, as before. Of course, $I_y \subseteq I$, because $I$ is an ideal in $A$. The minimality of $y$ implies that $x \prec \phi(x)$ for every $x \in I_y$. Note that $\phi(x) \prec \phi(y)$ and $\phi(x) \neq \phi(y)$ when $x \in I_y$, because $\phi$ is order-preserving and one-to-one by hypothesis. Thus $x \prec \phi(x) \prec \phi(y)$ for every $x \in I_y$. If $\phi(y) = x$ for some $x \in I_y$, then we would have that $x = \phi(x) = \phi(y)$, contradicting the fact that $\phi(x) \neq \phi(y)$. Hence $\phi(y) \notin I_y$, which says exactly that $y \prec \phi(y)$, contradicting the hypothesis that $\phi(y) \prec y$ and $\phi(y) \neq y$.

If $\phi(I)$ is also asked to be an ideal in $A$, then we can apply the previous argument to $\phi$ and to its inverse, to get that $\phi(x) = x$ for every $x \in I$ and thus $\phi(I) = I$. This case was discussed earlier, using a simpler but analogous argument.

Let $B$ be a subset of $A$, which can also be considered as a well-ordered set, using the restriction of the ordering $\prec$ from $A$ to $B$. Also let $\phi$ be a one-to-one order-preserving mapping from $A$ onto an ideal in $B$. More precisely, $\phi(A)$ is supposed to be an ideal in $B$ as a well-ordered set, but $\phi(A)$ may not be an ideal in $A$. If $\phi(A) \neq B$, then there is a $b \in B \setminus \phi(A)$, and $\phi(a) \prec b$ for every $a \in A$, because $\phi(A)$ is an ideal in $B$. In particular, $b \prec \phi(b)$, since $b$ is also an element of $A$. We also know that $x \prec \phi(x)$ for every $x \in A$, as before, so that $b \prec \phi(b)$. This implies that $\phi(b) = b$, contradicting the fact that $b \notin \phi(A)$. It follows that $\phi(A) = B$ when $\phi$ is a one-to-one order-preserving mapping from $A$ onto an ideal in $B \subseteq A$. This extends the case where $B = A$ that was mentioned before.

If $B$ is any well-ordered set, then either $B$ is isomorphic to an ideal in $A$, or $A$ is isomorphic to an ideal in $B$, as discussed previously. If $B$ is a subset of $A$, with the induced ordering, and $A$ is isomorphic to an ideal in $B$, then it follows that $A$ is isomorphic to $B$, as in the preceding paragraph. Thus we may conclude that every subset $B$ of $A$ is isomorphic to an ideal in $A$.

### 47 Some additional topics

#### 47.1 Products of finite sets

Let $X_1, X_2, \ldots$ be a sequence of finite sets, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product. More precisely, suppose that each $X_j$ is equipped with the discrete topology, and let $X$ be equipped with the corresponding product topology. We may as well suppose also that each $X_j$ has at least two elements, since $X = \emptyset$ when $X_j = \emptyset$ for any $j$, and the $X_j$’s with only one element do not really affect the topology on $X$. We have seen before that $X$ is compact, because it is sequentially compact and has a countable base for its topology. Let us now give another proof of this fact.

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $X$, and suppose for the sake of a contradiction that $X$ is not covered by finitely many $U_\alpha$’s. Of course, $X$ can be identified with the union of the sets

$$\{x_1\} \times \prod_{j=2}^{\infty} X_j$$

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with $x_1 \in X_1$. If each of these sets can be covered by only finitely many $U_{\alpha}$’s, then $X$ can be covered by only finitely many $U_{\alpha}$’s, since $X_1$ has only finitely many elements. Thus there is an $x_1 \in X_1$ such that (47.1) cannot be covered by finitely many $U_{\alpha}$’s. Similarly, there is an $x_2 \in X_2$ such that

\[(47.2)\quad \{x_1\} \times \{x_2\} \times \prod_{j=3}^{\infty} X_j\]

cannot be covered by finitely many $U_{\alpha}$’s. Continuing in this way, we get a sequence $x_1, x_2, x_3, \ldots$ of elements of $X_1, X_2, X_3, \ldots$, respectively, such that

\[(47.3)\quad \{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \prod_{j=n+1}^{\infty} X_j\]

cannot be covered by finitely many $U_{\alpha}$’s for any positive integer $n$. Let $x$ be the element of $X$ corresponding to the sequence $x_1, x_2, x_3, \ldots$. Because $\{U_{\alpha}\}_{\alpha \in A}$ covers $X$, there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. We also have that (47.3) is contained in $U_{\alpha_0}$ for some $n$, since $U_{\alpha_0}$ is an open set in $X$ with respect to the product topology. This contradicts the hypothesis that (47.3) cannot be covered by finitely many $U_{\alpha}$’s for any $n$, as desired.

In particular, one can apply this argument with $X_j = \{0, 1\}$ for each $j$, to get that the space $B$ of all binary sequences is compact with respect to the product topology. As before, the function that assigns to each binary sequence a real number in the usual way defines a continuous mapping from $B$ onto $[0, 1]$, and hence the compactness of $B$ then implies that $[0, 1]$ is compact. Of course, the preceding proof of the compactness of $X$ and thus $B$ is very similar to the standard proof of the compactness of $[0, 1]$, and so it is not surprising that one can use it to recover the compactness of $[0, 1]$. One can also show that $B$ is homeomorphic to Cantor’s middle-thirds set, which is reviewed next.

### 47.2 The Cantor set

Put $E_0 = [0, 1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$, and

\[(47.4)\quad E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].\]

In general, if $n$ is a nonnegative integer, then $E_n$ is a closed subset of $[0, 1]$ consisting of $2^n$ pairwise-disjoint closed intervals of length $3^{-n}$. To go from $E_n$ to $E_{n+1}$, one removes the open middle third of each of the intervals in $E_n$, as in the first two steps. Thus $E_{n+1} \subseteq E_n$ for each $n$, and

\[(47.5)\quad E = \bigcap_{n=0}^{\infty} E_n\]

is Cantor’s middle-thirds set. This is a closed subset of $[0, 1]$, which does not contain any interval in the real line of positive length.
Let $A_n$ be the set consisting of the endpoints of the $2^n$ intervals in $E_n$ for each $n \geq 0$. Thus $A_n$ has $2^{n+1}$ elements, $A_n \subseteq E_n$, and $A_n \subseteq A_{n+1}$ for each $n$. This implies that

\begin{equation}
A = \bigcup_{n=1}^{\infty} A_n
\end{equation}

is a countable set contained in the Cantor set $E$, and one can check that $A$ is dense in $E$. More precisely, if $x \in E_n$ for some $n \geq 0$, and if $I$ is the interval in $E_n$ of length $3^{-n}$ that contains $x$, then the distance from $x$ to the nearest endpoint of $I$ is less than or equal to one-half of $3^{-n}$.

As usual, every element of $[0, 1]$ can be represented by a binary expansion, and sometimes by two binary expansions. Similarly, every element of $[0, 1]$ has a ternary expansion, which is to say an expansion base 3, consisting of 0’s, 1’s, and 2’s. It is well known that the Cantor set consists exactly of elements of $[0, 1]$ with ternary expansions consisting of only 0’s and 2’s. Some elements of the Cantor set also have a ternary expansion that includes a 1, but ternary expansions that consist of only 0’s and 2’s are unique when they exist. Every ternary expansion determines an element of $[0, 1]$, and thus every ternary expansion with only 0’s and 2’s determines an element of the Cantor set.

If $b = \{b_j\}_{j=1}^{\infty}$ is a binary sequence, so that $b_j = 0$ or 1 for each $j$, then

\begin{equation}
f(b) = \sum_{j=1}^{\infty} 2 b_j 3^{-j}.
\end{equation}

is the element of $[0, 1]$ corresponding to the ternary expansion whose $j$th term is $2 b_j$. Of course, these ternary expansions contain only 0’s and 2’s, and every ternary expansion with only 0’s and 2’s comes from a unique binary sequence $b$ in this way. Thus $f$ defines a one-to-one mapping from the set $B$ of all binary sequences onto the Cantor set $E$, as in the previous paragraph. One can check that $f$ is a homeomorphism from $B$ as a product of a sequence of copies of $\{0, 1\}$ with the product topology associated to the discrete topology on $\{0, 1\}$ onto the Cantor set with the topology induced by the standard topology on the real line.

### 47.3 Quotient spaces and mappings

Let $X$ be a set, and let $\sim$ be an equivalence relation on $X$. Also let $X/\sim$ be the set of equivalence classes in $X$ determined by $\sim$, and let $\pi$ be the natural quotient mapping from $X$ onto $X/\sim$, which sends each point $x \in X$ to the equivalence class in $X$ that contains $x$. If $X$ is a topological space, then the \textit{quotient topology} is defined on $X/\sim$ by saying that a set $W \subseteq X/\sim$ is an open set if and only if $\pi^{-1}(W)$ is an open set in $X$. It is easy to see that this defines a topology on $X$, using the fact that

\begin{equation}
\pi^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} \pi^{-1}(E_\alpha)
\end{equation}

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and
\[ (47.9) \quad \pi^{-1}\left( \bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} \pi^{-1}(E_\alpha) \]

for any family \(\{E_\alpha\}_{\alpha \in A}\) of subsets of \(X/\sim\).

Similarly, let \(X\) and \(Y\) be topological spaces, and let \(f\) be a mapping from \(X\) onto \(Y\). We say that \(f\) is a quotient mapping if \(V \subseteq Y\) is an open set if and only if \(f^{-1}(V)\) is an open set in \(X\). Equivalently, \(f\) is a quotient mapping if \(f\) is continuous and \(V \subseteq Y\) is an open set whenever \(f^{-1}(V)\) is an open set in \(X\). Note that a one-to-one quotient mapping is the same as a homeomorphism.

If \(X\) is a topological space, \(\sim\) is an equivalence relation on \(X\), and \(\pi\) is the canonical quotient mapping from \(X\) onto the corresponding quotient space \(X/\sim\), then \(\pi\) is a quotient mapping with respect to the quotient topology on \(X/\sim\), by construction. Conversely, let \(X\) and \(Y\) be topological spaces, and let \(f\) be a mapping from \(X\) onto \(Y\). Consider the relation on \(X\) defined by \(x \sim x'\) when \(f(x) = f(x')\). It is easy to see that this defines an equivalence relation on \(X\), for which the equivalence classes are the subsets of \(X\) of the form \(f^{-1}(y)\) for some \(y \in Y\). This permits one to identify \(X/\sim\) with \(Y\), and \(\pi\) with \(f\), so that \(f\) is a quotient mapping if and only if the topology on \(Y\) corresponds to the quotient topology on \(X/\sim\).

Let \(X\) and \(Y\) be topological spaces again, and remember that
\[ (47.10) \quad X \setminus f^{-1}(E) = f^{-1}(Y \setminus E) \]

for any mapping \(f : X \to Y\) and subset \(E\) of \(Y\). Using this, one can check that a mapping \(f\) from \(X\) onto \(Y\) is a quotient mapping if and only if it has the property that \(E \subseteq Y\) is a closed set if and only if \(f^{-1}(E)\) is a closed set in \(X\).

As before, this is equivalent to asking that \(f\) be continuous, and that \(E \subseteq Y\) be a closed set whenever \(f^{-1}(E)\) is a closed set in \(X\).

Observe that \(f(f^{-1}(V)) \subseteq V\) for every mapping \(f : X \to Y\) and subset \(V\) of \(Y\), and that \(f(f^{-1}(V)) = V\) when \(f\) maps \(X\) onto \(Y\). This implies that a continuous mapping from \(X\) onto \(Y\) is a quotient mapping if it sends open subsets of \(X\) to open subsets of \(Y\), or if it sends closed subsets of \(X\) to closed subsets of \(Y\).

Suppose that \(f\) is a continuous mapping from \(X\) onto \(Y\), that \(X\) is compact, and that \(Y\) is Hausdorff. Thus closed subsets of \(X\) are also compact, and hence are sent to compact subsets of \(Y\), because \(f\) is continuous. We also know that compact subsets of \(Y\) are closed sets, because \(Y\) is Hausdorff. This shows that \(f\) maps closed subsets of \(X\) to closed subsets of \(Y\) under these conditions, so that \(f\) is a quotient mapping, as in the previous paragraph. This was mentioned previously in the special case where \(f\) is also one-to-one, so that \(f\) is a homeomorphism.

Remember that a topological space \(Y\) satisfies the first separation condition if and only if \(\{y\}\) is a closed set in \(Y\) for every \(y \in Y\). If \(f : X \to Y\) is a quotient mapping, then it follows that \(Y\) satisfies the first separation condition if and only if \(f^{-1}(y)\) is a closed set in \(X\) for every \(y \in Y\).
47.4 Homotopic paths

Let $X$ be a topological space, and let $p, q$ be elements of $X$. Also let $f_0, f_1$ be continuous mappings from $[0, 1]$ into $X$ such that $f_0(0) = f_1(0) = p$ and $f_0(1) = f_1(1) = q$. We say that $f_0$ and $f_1$ are homotopic as continuous paths in $X$ from $p$ to $q$ if there is a continuous mapping $F$ from $[0, 1] \times [0, 1]$ into $X$ such that

$$F(0, t) = p \text{ and } F(1, t) = q$$

for every $t \in [0, 1]$, and

$$F(r, 0) = f_0(r) \text{ and } F(r, 1) = f_1(r)$$

for every $r \in [0, 1]$. Thus $f_t(r) = F(r, t)$ is a continuous path in $X$ from $p$ to $q$ as a function of $r \in [0, 1]$ for each $t \in [0, 1]$, which reduces to the given paths $f_0(r), f_1(r)$ when $t = 0, 1$. The function $F(r, t)$ defines a homotopy between $f_0(r)$ and $f_1(r)$, and the continuity of this function on $[0, 1] \times [0, 1]$ basically means that we have a continuous family of paths from $p$ to $q$ in $X$.

If $X = \mathbb{R}^n$ with the standard topology for some positive integer $n$, then

$$F(r, t) = (1 - t)f_0(r) + tf_1(t)$$

defines a homotopy between $f_0(r)$ and $f_1(r)$. Hence every pair of paths in $\mathbb{R}^n$ with the same endpoints are homotopic. Similarly, if $X$ is a convex set in $\mathbb{R}^n$, with the topology induced by the standard topology on $\mathbb{R}^n$, then every pair of paths in $X$ with the same endpoints are homotopic, using the same homotopy (47.13). However, if $X = \mathbb{R}^2 \setminus \{(0, 0)\}$, with the topology induced by the standard topology on $\mathbb{R}^2$, then it can be shown that different paths in $X$ with the same endpoints may not be homotopic in $X$. Basically, this is because different paths in $X$ may wrap around $(0, 0)$ in different ways, even if $(0, 0)$ is not included in $X$.

As another class of examples, let $X$ be any topological space again, and let $f$ be a continuous mapping of $[0, 1]$ into $X$. Also let $a$ be a continuous mapping from $[0, 1]$ into $[0, 1]$ such that $a(0) = 0$ and $a(1) = 1$. Thus $a$ maps $[0, 1]$ onto $[0, 1]$, by the intermediate value theorem. In this case,

$$F(r, t) = f((1 - t)r + ta(r))$$

defines a homotopy between $f(r)$ and $f(a(r))$. In effect, this uses the convexity of $[0, 1]$, instead of the convexity of $X$. This shows that any continuous path $f$ in $X$ is homotopic to any continuous reparameterization of $f$.

Let $X$ be any topological space, and let $p, q$ be elements of $X$, as before. If $f$ is a continuous mapping from $[0, 1]$ into $X$ such that $f(0) = p$ and $f(1) = q$, then $f$ is obviously homotopic to itself, because $F(r, t) = f(r)$ defines a homotopy between $f$ and itself. Similarly, if $f_0, f_1$ are continuous mappings from $[0, 1]$ into $X$ such that $f_0(0) = f_1(0) = p$, $f_0(1) = f_1(1) = q$, and $F(r, t)$ is a homotopy between $f_0$ and $f_1$, then $F(r, 1 - t)$ defines a homotopy between $f_1$ and $f_0$. Thus the property of being homotopic is symmetric in the two paths. If $f_2$ is
another continuous mapping from \([0, 1]\) into \(X\) such that \(f_2(0) = p, f_2(1) = q,\)
and \(f_2\) is homotopic to \(f_1,\) then it is not too difficult to show that \(f_0\) is also homotopic to \(f_2.\) Basically, one simply continues a homotopy from \(f_0\) to \(f_1\) with a homotopy from \(f_1\) to \(f_2,\) using a change of variables to get a homotopy from \(f_0\) to \(f_2\) defined on \([0, 1] \times [0, 1].\) This implies that the property of being homotopic defines an equivalence relation on the set of continuous paths in \(X\) from \(p\) to \(q.\) The corresponding equivalence classes are known as homotopy classes.

### 47.5 The fundamental group

Let \(X\) be a topological space, and let \(p\) be an element of \(X.\) A continuous mapping \(f : [0, 1] \to X\) such that \(f(0) = f(1) = p\) is called a continuous loop based at \(p.\) If \(g : [0, 1] \to X\) is another continuous loop based at \(p,\) then we can combine \(f\) and \(g\) to get a continuous loop \(f \cdot g\) based at \(p.\) More precisely, this can be defined by

\[
(f \cdot g)(r) = \begin{cases} f(2r) & \text{when } 0 \leq r \leq 1/2 \\ g(2r - 1) & \text{when } 1/2 \leq r \leq 1. \end{cases}
\]

If \(f_0, f_1, g_0,\) and \(g_1\) are continuous loops based at \(p\) such that \(f_0\) is homotopic to \(f_1\) and \(g_0\) is homotopic to \(g_1,\) then it is easy to check that \(f_0 \cdot g_0\) is homotopic to \(f_1 \cdot g_1,\) by combining homotopies between \(f_0\) and \(f_1\) and between \(g_0\) and \(g_1,\) in the same way that individual loops are combined to get a homotopy between \(f_0 \cdot g_0\) and \(f_1 \cdot g_1.\)

If \(f\) is a continuous loop in \(X\) based at \(p,\) then let \([f]\) be the homotopy class of continuous loops based at \(p\) that contains \(f,\) which is the collection of all continuous loops in \(X\) based at \(p\) that are homotopic to \(f.\) If \(g\) is another continuous loop in \(X\) based at \(p,\) then put

\[
([f] \cdot [g]) = [f \cdot g].
\]

If \(f', g'\) are continuous loops in \(X\) based at \(p\) that are homotopic to \(f, g,\) respectively, then \(f' \cdot g'\) is homotopic to \(f \cdot g,\) as in the preceding paragraph, and hence

\[
([f'] \cdot [g']) = [f \cdot g].
\]

This implies that \([f] \cdot [g]\) is a well-defined operation on homotopy classes of continuous loops in \(X\) based at \(p,\) since it does not depend on the choice of representatives of the homotopy classes \([f]\) and \([g].\)

If \(f, g,\) and \(h\) are continuous loops in \(X\) based at \(p,\) then \((f \cdot g) \cdot h\) and \(f \cdot (g \cdot h)\) are also continuous loops in \(X\) based at \(p,\) which are not normally quite the same. However, it is easy to see that \((f \cdot g) \cdot h\) and \(f \cdot (g \cdot h)\) are reparameterizations of each other, and hence they are homotopic to each other. This implies that

\[
([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h]),
\]

so that the corresponding operation on homotopy classes of continuous loops in \(X\) based at \(p\) is associative.
Let \( c_p : [0, 1] \to X \) be the constant loop based at \( p \), defined by \( c_p(r) = p \) for each \( r \in [0, 1] \). If \( f \) is any continuous loop in \( X \) based at \( p \), then it is easy to see that \( c_p \cdot f \) and \( f \cdot c_p \) can both be given as reparameterizations of \( f \). Thus \( c_p \cdot f \) and \( f \cdot c_p \) are both homotopic to \( f \), so that

\[
[c_p] \cdot [f] = [f] \cdot [c_p] = [f].
\]

This shows that \([c_p]\) plays the role of the identity element for this operation on homotopy classes of continuous loops in \( X \) based at \( p \).

If \( f \) is a continuous loop in \( X \) based at \( p \), then let \( \tilde{f} \) be the continuous loop in \( X \) based at \( p \) defined by

\[
\tilde{f}(r) = f(1 - r).
\]

If \( f' \) is another continuous loop in \( X \) based at \( p \) which is homotopic to \( f \), then it is easy to see that \( \tilde{f}' \) is homotopic to \( \tilde{f} \), by replacing \( r \) with \( 1 - r \) in the homotopy between \( f \) and \( f' \). Thus the homotopy class \([f]\) depends only on the homotopy class \([f']\), and is denoted \([f]^{-1}\). One can also check that \( f \cdot \tilde{f} \) and \( \tilde{f} \cdot f \) are both homotopic to \( c_p \), as continuous loops in \( X \) based at \( p \), so that

\[
[f]^{-1} \cdot [f] = [f] \cdot [f]^{-1} = [c_p].
\]

It follows that the collection of homotopy classes of continuous loops in \( X \) based at \( p \) forms a group with respect to the operation discussed in the previous paragraphs, known as the fundamental group of \( X \) at \( p \). If \( q \) is another element of \( X \) and \( a_{p,q} \) is a continuous path in \( X \) from \( p \) to \( q \), then one can show that the fundamental group of \( X \) at \( p \) is isomorphic to the fundamental group of \( X \) at \( q \). The basic idea is to map continuous loops in \( X \) based at \( p \) to continuous loops in \( X \) based at \( q \), using \( a_{p,q} \) to go back and forth between \( p \) and \( q \). It is easy to see that this mapping sends homotopic loops at \( p \) to homotopic loops at \( q \), and thus leads to a mapping from homotopy classes of loops at \( p \) to homotopy classes of loops at \( q \). One can also check that this mapping is compatible with the operations on homotopy classes of loops discussed in the previous paragraphs, so that it defines a homomorphism from the fundamental group of \( X \) at \( p \) into the fundamental group of \( X \) at \( q \). There is an analogous mapping from continuous loops in \( X \) at \( q \) to continuous loops in \( X \) at \( p \), for which the corresponding mapping between homotopy classes is the inverse of the preceding one. If \( X \) is pathwise connected, then the fundamental group of \( X \) at \( p \) is isomorphic to the fundamental group of \( X \) at \( q \) for every \( p, q \in X \).

If \( X = \mathbb{R}^n \) with the standard topology for some positive integer \( n \), then the fundamental group of \( X \) is trivial at every point, because any two continuous loops based at the same point are homotopic to each other, as before. Similarly, if \( X \) is a convex set in \( \mathbb{R}^n \) equipped with the topology induced by the standard topology on \( \mathbb{R}^n \), then the fundamental group of \( X \) is trivial at every point. If \( X = \mathbb{R}^2 \setminus \{(0,0)\} \), then it can be shown that the fundamental group of \( X \) at any point is isomorphic to the group \( \mathbb{Z} \) of integers with respect to addition. The isomorphism is given by the winding number of a loop in \( \mathbb{R}^2 \setminus \{(0,0)\} \), which counts the number of times that such a loop wraps around the origin, also taking orientations into account.

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47.6 Topological groups

Let $G$ be a group. Thus $G$ is a set equipped with a binary operation, which can be expressed multiplicatively by $xy$ for $x, y \in G$. This operation is supposed to satisfy the associative law

$$\tag{47.22} (xy)z = x(yz)$$

for every $x, y, z \in G$, and there should also be an identity element $e$ in $G$ such that

$$\tag{47.23} ex = xe = x$$

for every $x \in G$. Each $x \in G$ is supposed to have an inverse $x^{-1} \in G$ that satisfies

$$\tag{47.24} x x^{-1} = x^{-1} x = e,$$

and which is uniquely determined by $x$.

In order for $G$ to be a topological group, $G$ should also be equipped with a topology for which the group operations are continuous. More precisely, this means that

$$\tag{47.25} (x, y) \mapsto xy$$

should be continuous as a mapping from $G \times G$ into $G$, using the product topology on $G \times G$ associated to the given topology on $G$. Similarly,

$$\tag{47.26} x \mapsto x^{-1}$$

should be continuous as a mapping from $G$ to itself. This should actually be a homeomorphism from $G$ onto itself, because $(x^{-1})^{-1} = x$ for each $x \in G$, which implies that this mapping is its own inverse.

If $a, b \in G$, then

$$\tag{47.27} L_a(x) = ax, \quad R_b(x) = xb$$

define mappings from $G$ into itself, which are the left and right translation mappings corresponding to $a, b$, respectively. Continuity of multiplication in the group implies that $L_a$ and $R_b$ are continuous mappings on $G$ for each $a, b \in G$. Note that $L_a$ and $R_b$ are invertible as mappings on $G$ for each $a, b \in G$, with inverses given by $L_{a^{-1}}, R_{b^{-1}}$, respectively. Thus $L_a$ and $R_b$ are actually homeomorphisms from $G$ onto itself for each $a, b \in G$, since $L_{a^{-1}}$ and $R_{b^{-1}}$ are also continuous.

If $a, b \in G$ and $A, B \subseteq G$, then put

$$\tag{47.28} aB = L_a(B), \quad Ab = R_b(A),$$

and

$$\tag{47.29} AB = \{xy : x \in A, y \in B\}.$$
previous paragraph. Similarly, if $A$ and $B$ are closed subsets of $G$, then $aB$ and $Ab$ are closed sets for every $a, b \in G$. Observe that

$$AB = \bigcup_{a \in A} aB = \bigcup_{b \in B} Ab$$

for every $A, B \subseteq G$. If $A$ or $B$ is an open set in $G$, then it follows that $AB$ is an open set in $G$, because it is a union of open subsets of $G$. Let us also put

$$A^{-1} = \{a^{-1} : a \in A\}.$$

If $A$ is an open or closed set in $G$, then $A^{-1}$ has the same property, because $x \mapsto x^{-1}$ is a homeomorphism on $G$.

It is customary to also ask that $\{e\}$ be a closed set in a topological group $G$. This implies that $\{a\}$ is a closed set in $G$ for every $a \in G$, and hence that $G$ satisfies the first separation condition, because of continuity of translations, as before. One can also show that $G$ is Hausdorff and even regular, as follows.

Suppose that $E$ is a closed subset of $G$ that does not contain $e$. Thus $G \setminus E$ is an open set that contains $e$. This implies that there are open subsets $U, V$ of $G$ containing $e$ such that

$$UV^{-1} \subseteq G \setminus E,$$

because of the continuity of the group operations. Equivalently, this means that $UV^{-1} \cap E = \emptyset$, which implies that

$$U \cap EV = \emptyset.$$

Note that $E \subseteq EV$, since $e \in V$, and that $EV$ is an open set, because $V$ is open. If instead $E_1$ is a closed set in $G$ that does not contain a particular element $x \in G$, then one can apply the preceding argument to $E = x^{-1}E_1$. If $U, V$ are as before, then $xU$ is an open set in $G$ that contains $x$, and which is disjoint from the open set $xEV = E_1V$ that contains $V$. This shows that $G$ is regular, and hence Hausdorff, as desired.

Of course, any group is a topological group with respect to the discrete topology. The real line is also a topological group with respect to addition and the standard topology.

If $G$ is a topological group, and if $G$ satisfies the local countability condition at $e$, then $G$ satisfies the local countability condition at each point, because of continuity of translations. A well-known theorem states that the topology on $G$ is determined by a metric under these conditions.

47.7 Topological vector spaces

Let $V$ be a vector space over the real numbers. Thus $V$ is set equipped with operations of addition and scalar multiplication that satisfy certain conditions. In order for $V$ to be a topological vector space, $V$ should also be equipped with a topology for which the vector space operations are continuous. More precisely,
this means that addition should be continuous as a mapping from $V \times V$ into $V$. Similarly, scalar multiplication should be continuous as a mapping from $\mathbb{R} \times V$ into $V$, using the product topology on $\mathbb{R} \times V$ associated to the standard topology on $\mathbb{R}$ and the given topology on $V$.

Of course, a vector space $V$ is an abelian group with respect to addition. The additive inverse $-v$ of a vector $v \in V$ is equal to $-1 \in \mathbb{R}$ times $v$ in the sense of scalar multiplication. Thus the continuity of $v \mapsto -v$ follows from the continuity of scalar multiplication in this case. As for topological groups, it is customary to ask that $\{0\}$ be a closed set in a topological vector space $V$, where $0$ is the additive identity element in $V$. This implies that $V$ is Hausdorff and even regular, as before.

A nonnegative real-valued function $\|v\|$ on a vector space $V$ over the real numbers is said to be a norm if $\|v\| = 0$ if and only if $v = 0$, 
\begin{equation}
\|tv\| = |t| \|v\|
\end{equation}
for every $t \in \mathbb{R}$ and $v \in V$, and 
\begin{equation}
\|v + w\| \leq \|v\| + \|w\|
\end{equation}
for every $v, w \in V$. Here $|t|$ is the absolute value of a real number $t$. In this case, it is easy to see that 
\begin{equation}
d(v, w) = \|v - w\|
\end{equation}
defines a metric on $V$. One can also check that $V$ is a topological vector space with respect to the topology determined by this metric.

If $n$ is a positive integer, then $\mathbb{R}^n$ is a topological vector space, where the vector space operations are defined coordinatewise in the usual way, and $\mathbb{R}^n$ is equipped with the standard topology. The standard topology on $\mathbb{R}^n$ is the same as the topology associated to the standard Euclidean metric, which is the metric associate to the standard Euclidean norm on $\mathbb{R}^n$ as in the previous paragraph. More generally, any product of copies of $\mathbb{R}$ is a topological vector space, where the vector space operations are again defined coordinatewise, and one uses the product topology associated to the standard topology on $\mathbb{R}$. However, if the space is the product of infinitely many copies of $\mathbb{R}$, then one can show that this topology is not determined by a norm.

### 47.8 Uniform spaces

If $X$ is a topological space, then many of the same notions related to continuity and convergence can be defined as on a metric space. However, this does not include Cauchy sequences or uniform continuity. Indeed, one can give examples of metrics $d_1(x, y)$ and $d_2(x, y)$ on a set $X$ that determine the same topology on $X$, but not the same Cauchy sequences or uniformly continuous mappings. Of course, this is closely related to the fact that a homeomorphism between metric spaces may not be uniformly continuous, and that its inverse may not be uniformly continuous.
To deal with this, there is an abstract notion of a uniform structure on a set $X$. This involves subsets of the product $X \times X$ that contain the diagonal $\Delta$, which is the set of $(x, y) \in X \times X$ with $x = y$. Basically, these subsets of $X \times X$ are supposed to describe uniform families of neighborhoods of elements of $X$, but we shall not go into the details here. If $X$ is equipped with a metric $d(x, y)$, for instance, then one can consider subsets of $X \times X$ containing all $(x, y)$ with $X, y \in X$ and $d(x, y) < \epsilon$ for some $\epsilon > 0$.

Topological groups provide another interesting special case of this, which includes topological vector spaces in particular. Let $G$ be a topological group, and let $U$ be an open set in $G$ that contains the identity element $e$. This leads to an interesting subset of $G \times G$ that contains the diagonal, consisting of all $(x, y)$ with $x, y \in G$ such that $xy^{-1} \in U$. Alternatively, one could look at the set of all $(x, y)$ with $x, y \in G$ such that $y^{-1}x \in U$. This is the same as the previous set when $G$ is abelian, but otherwise the difference could be significant. One could also consider the set of $(x, y)$ with $x, y \in G$ such that $(xy^{-1})^{-1} = y^{-1}x \in U$, which amounts to replacing $U$ with $U^{-1}$. This is not too serious, because of the continuity of $x \mapsto x^{-1}$ on $G$. Similarly, the condition $(y^{-1}x)^{-1} = x^{-1}y \in U$ is the same as $y^{-1}x \in U^{-1}$.

### 47.9 The supremum norm

Let $X$ be a (nonempty) topological space, and let $C_b(X)$ be the space of bounded continuous real-valued functions on $X$. More precisely, a real-valued function $f$ on $X$ is said to be bounded on $X$ if $f(X)$ is a bounded subset of the real line, so that $|f(x)| \leq A$ for some nonnegative real number $A$ and every $x \in X$. If $X$ is compact, then $f(X)$ is a compact set in $\mathbb{R}$ for every continuous real-valued function $f$ on $X$, and hence $f(X)$ is automatically bounded in $\mathbb{R}$. It is easy to see that $C_b(X)$ is a vector space over the real numbers with respect to pointwise addition and scalar multiplication, which may be considered as a linear subspace of the vector space $C(X)$ of all continuous real-valued functions on $X$. If $C(X)$ separates points on $X$, then $C_b(X)$ separates points on $X$ too, because one can get take maxima and minima of arbitrary continuous real-valued functions on $X$ with constants to get bounded continuous functions on $X$. Of course, constant functions on any topological space are bounded and continuous.

If $f \in C_b(X)$, then the supremum norm of $f$ on $X$ is defined by

\begin{equation}
\|f\| = \sup_{x \in X} |f(x)|.
\end{equation}

It is easy to check that this is indeed a norm on $C_b(X)$ of $X$ as a vector space over the real numbers, which determines the corresponding supremum metric

\begin{equation}
\|f - g\|
\end{equation}

on $C_b(X)$. As usual, a sequence $\{f_j\}_{j=1}^\infty$ of bounded continuous real-valued functions on $X$ converges to another bounded continuous real-valued function $f$ on $X$ with respect to the supremum metric if and only if $\{f_j\}_{j=1}^\infty$ converges...
to \( f \) uniformly on \( X \). If a sequence \( \{f_j\}_{j=1}^\infty \) of bounded continuous real-valued functions on \( X \) converges to a real-valued function \( f \) uniformly on \( X \), then \( f \) is bounded and continuous on \( X \), by standard arguments. Similarly, one can show that \( \mathcal{C}_b(X) \) is complete as a metric space with respect to the supremum metric, in the sense that every Cauchy sequence in \( \mathcal{C}_b(X) \) converges to an element of \( \mathcal{C}_b(X) \). If \( \{f_j\}_{j=1}^\infty \) is a Cauchy sequence in \( \mathcal{C}_b(X) \) with respect to the supremum metric, then \( \{f_j(x)\}_{j=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \) for each \( x \in X \), which implies that \( \{f_j(x)\}_{j=1}^\infty \) converges in \( \mathbb{R} \) for each \( x \in X \), because the real line is complete with respect to the standard metric. This shows that \( \{f_j\}_{j=1}^\infty \) converges to a real-valued function \( f \) pointwise on \( X \), and one can then use the Cauchy condition with respect to the supremum metric to check that \( \{f_j\}_{j=1}^\infty \) converges to \( f \) uniformly on \( X \), which implies that \( f \) is bounded and continuous on \( X \), as before.

Suppose now that \( X \) is a locally compact Hausdorff topological space which is not compact. Remember that a continuous real-valued function \( f \) on \( X \) is said to vanish at infinity on \( X \) if for each \( \epsilon > 0 \) there is a compact set \( K_\epsilon \subseteq X \) such that \( |f(x)| < \epsilon \) for every \( x \in X \setminus K_\epsilon \). This implies that \( f \) is bounded on \( X \), since any continuous function on \( X \) is bounded on compact subsets of \( X \). Thus the space \( \mathcal{C}_b(X) \) of continuous real-valued functions on \( X \) that vanish at infinity is contained in \( \mathcal{C}_0(X) \), and in fact \( \mathcal{C}_0(X) \) is a linear subspace of \( \mathcal{C}_b(X) \). It is not too difficult to show that \( \mathcal{C}_0(X) \) is also closed with respect to the supremum metric on \( \mathcal{C}_b(X) \).

Remember that the support of a continuous real-valued function \( f \) on \( X \) is defined to be the closure of the set of \( x \in X \) such that \( f(x) \neq 0 \), and that the space of continuous real-valued functions with compact support on \( X \) is denoted \( \mathcal{C}_{\text{com}}(X) \). This is a linear subspace of \( \mathcal{C}_0(X) \), and it is not too difficult to show that \( \mathcal{C}_{\text{com}}(X) \) is dense in \( \mathcal{C}_0(X) \) with respect to the supremum metric. To see this, let \( f \in \mathcal{C}_0(X) \) and \( \epsilon > 0 \) be given, and let \( K_\epsilon \) be a compact set in \( X \) such that \( |f(x)| < \epsilon \) for every \( x \in X \setminus K_\epsilon \), as in the preceding paragraph. Using Urysohn's lemma, one can get a continuous real-valued function \( \theta_\epsilon \) on \( X \) such that \( \theta_\epsilon(x) = 1 \) for every \( x \in K_\epsilon \), \( \theta_\epsilon(x) = 0 \) for every \( x \in X \), and \( \theta_\epsilon \) has compact support on \( X \). Thus \( \theta_\epsilon f \) has compact support on \( X \), and

\[
|f(x) - \theta_\epsilon(x) f(x)| = (1 - \theta_\epsilon(x)) |f(x)| < \epsilon
\]

for every \( x \in X \), as desired.

### 47.10 Uniform continuity

Let \( (M, d(x,y)) \) and \( (N, \rho(u,v)) \) be metric spaces. As usual, a mapping \( f \) from \( M \) into \( N \) is said to be uniformly continuous if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
\rho(f(x), f(y)) < \epsilon
\]

for every \( x, y \in M \) such that \( d(x,y) < \delta \). Of course, every uniformly continuous mapping from \( M \) into \( N \) is continuous, and conversely it is well known that every continuous mapping from \( M \) into \( N \) is uniformly continuous when \( M \)
is compact. As mentioned earlier, uniform continuity of mappings between topological spaces does not make sense without some additional structure, but there is a special case where a type of uniform continuity condition is natural, that we shall described next.

Let $X$, $Y$ be topological spaces, and let $(N, \rho(u, v))$ be a metric space again. Also let $f(x, y)$ be a function on $X \times Y$ with values in $N$. In particular, $N$ could be the real line with the standard metric, so that $f(x, y)$ is a real-valued function on $X \times Y$. Let us say that $f(x, y)$ is continuous in $x$ at a point $x_0 \in X$ uniformly in $y$ if for every $\varepsilon > 0$ there is an open set $U$ of $X$ such that $x_0 \in U$ such that

$$\rho(f(x, y), f(x_0, y)) < \varepsilon$$

for every $x \in U$ and $y \in Y$. If this condition holds for every $x_0 \in X$, then we say that $f(x, y)$ is continuous in $x$ uniformly in $y$.

If $f(x, y)$ is continuous in $x$ at $x_0 \in X$ uniformly in $y$, then $f(x, y)$ is obviously continuous as a function of $x$ at $x_0$ for each $y \in Y$. In the other direction, let us show that $f(x, y)$ is continuous in $x$ at $x_0$ uniformly in $y$ when $f(x, y)$ is continuous as a function on $X \times Y$ with the product topology at $(x_0, y)$ for every $y \in Y$ and $Y$ is compact. This implies that $f(x, y)$ is continuous in $x$ uniformly in $y$ when $f(x, y)$ is continuous on $X \times Y$ with respect to the product topology and $Y$ is compact.

Let $\varepsilon > 0$ be given, and for each $y \in Y$, let $W(y)$ be an open set in $X \times Y$ such that

$$\rho(f(x, z), f(x_0, z)) < \frac{\varepsilon}{2}$$

for every $(x, z) \in W(y)$. By definition of the product topology on $X \times Y$, there are open subsets $U(y), V(y)$ of $X, Y$, respectively, such that $x_0 \in U(y), y \in V(y)$, and

$$U(y) \times V(y) \subseteq W(y).$$

If $x \in U(y)$ and $z \in V(y)$, then we can apply (47.42) to $(x, z)$ and to $(x_0, z)$ to get that

$$\rho(f(x, z), f(x_0, z)) \leq \rho(f(x, z), f(x_0, y)) + \rho(f(x_0, y), f(x_0, z)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $Y$ is compact, then there are finitely many elements $y_1, \ldots, y_n$ of $Y$ such that

$$Y = \bigcup_{j=1}^{n} V(y_j).$$

In this case,

$$U = \bigcap_{j=1}^{n} U(y_j)$$

is an open set in $X$ that contains $x_0$, and it is easy to see that (47.44) holds for every $x \in U$ and $z \in Y$, as desired.
47.11 The supremum metric

Let $Y$ be a topological space, and let $(N, \rho(u, v))$ be a metric space. Remember that a subset of $N$ is said to be bounded if it is contained in a ball of finite radius, and that a mapping $f : Y \to N$ is bounded when $f(Y)$ is a bounded set in $N$. Let $C_b(Y, N)$ be the space of bounded continuous mappings from $Y$ into $N$. If $Y$ is compact and $f : Y \to N$ is continuous, then $f(Y)$ is compact and hence bounded in $N$, and $C_b(Y, N)$ reduces to the space $C(Y, N)$ of all continuous mappings from $Y$ into $N$. If $N$ is the real line with the standard metric, then $C_b(Y, N)$ is the same as the vector space $C_0(Y)$ of bounded continuous real-valued functions on $Y$, discussed earlier.

If $f, g \in C_b(Y, N)$, then put

$$(47.47) \quad \theta(f, g) = \sup_{y \in Y} \rho(f(y), g(y)).$$

One can check that this defines a metric on $C_b(Y, N)$, known as the supremum metric. If $N = \mathbb{R}$ with the standard metric, then this is the same as the metric associated to the supremum norm on $C_b(Y)$. As before, a sequence $\{f_j\}_{j=1}^\infty$ of bounded continuous mappings from $Y$ into $N$ converges to another bounded continuous mapping $f$ from $Y$ into $N$ with respect to the supremum metric if and only if $\{f_j\}_{j=1}^\infty$ converges to $f$ uniformly on $Y$. If $\{f_j\}_{j=1}^\infty$ is a sequence of bounded continuous mappings from $Y$ into $N$ that converges uniformly to a mapping $f : Y \to N$, then $f$ is also bounded and continuous, by standard arguments. If $N$ is complete, then one can check that $C_b(Y, N)$ is complete with respect to the supremum metric as well. More precisely, completeness of $N$ implies that a Cauchy sequence $\{f_j\}_{j=1}^\infty$ in $C_b(Y, N)$ with respect to the supremum metric converges pointwise to a mapping $f : Y \to N$, and one can then show that the Cauchy condition with respect to the supremum metric implies that $\{f_j\}_{j=1}^\infty$ converges to $f$ uniformly on $Y$.

Let $X$ be another topological space, and suppose that $\Phi$ is a continuous mapping from $X$ into $C_b(Y, N)$, with respect to the supremum metric on $C_b(Y, N)$. If $x \in X$, then $\Phi(x)$ is a bounded continuous mapping from $Y$ into $N$, which can then be evaluated at $y \in Y$ to get an element $\Phi(x)(y)$ of $N$. Thus $\Phi$ determines a mapping $\phi$ from $X \times Y$ into $N$, where

$$(47.48) \quad \phi(x, y) = \Phi(x)(y)$$

for every $x \in X$ and $y \in Y$. The hypothesis that $\Phi$ map $X$ into $C_b(Y, N)$ means that $\phi(x, y)$ is a bounded continuous function of $y \in Y$ for each $x \in X$. The condition that $\Phi$ be a continuous mapping from $X$ into $C_b(Y, N)$ with respect to the supremum metric is then equivalent to asking that $\phi(x, y)$ be continuous in $x$ uniformly in $y$.

If $\phi : X \times Y \to N$ has the properties that $\phi(x_0, y)$ is continuous as a function of $y \in Y$ for some $x_0 \in X$, and that $\phi(x, y)$ is continuous as a function of $x$ at $x_0$ uniformly in $y$, then it is easy to see that $\phi$ is continuous as a function on $X \times Y$ with respect to the product topology at $(x_0, y)$ for each $y \in Y$. In particular, if $\Phi$ is a continuous mapping from $X$ into $C_b(Y, N)$, as in the preceding paragraph,
then the corresponding function $\phi(x, y)$ is continuous as a mapping from $X \times Y$ with the product topology into $N$. Conversely, if $\phi$ is a continuous mapping from $X \times Y$ into $N$ and $Y$ is compact, then $\phi(x, y)$ is bounded and continuous as a function of $y \in Y$ for each $x \in X$, and hence $\phi(x, y)$ determines a mapping $\Phi$ from $X$ into $\mathcal{C}(Y, N)$ as in (47.48). We have seen before that $\phi(x, y)$ is also continuous in $x$ uniformly in $y$ under these conditions, so that $\Phi$ is continuous with respect to the supremum metric on $\mathcal{C}(Y, N)$.

47.12 An approximation theorem

Let $X$ and $Y$ be compact Hausdorff topological spaces, and let $f(x, y)$ be a continuous real-valued function on $X \times Y$, with respect to the product topology. We would like to show that $f(x, y)$ can be approximated by finite sums of products of continuous functions on $X$ and on $Y$, uniformly on $X \times Y$.

Let $\epsilon > 0$ be given. Because $f(x, y)$ is continuous on $X \times Y$ and $Y$ is compact, we have seen previously that $f(x, y)$ is continuous in $x$ uniformly in $y$. This implies that for each $x \in X$ there is an open set $U_1(x) \subseteq X$ such that $x \in U_1(x)$ and

$$|f(w, y) - f(x, y)| < \epsilon$$  \hfill (47.49)

for every $w \in U_1(x)$ and $y \in Y$. Using Urysohn’s lemma, for each $x \in X$ we can get a continuous nonnegative real-valued function $\phi_x$ on $X$ such that $\phi_x(x) > 0$ and $\phi_x(w) = 0$ when $w \in X \setminus U_1(x)$. Put

$$U_2(x) = \{w \in W : \phi_x(w) > 0\},$$  \hfill (47.50)

so that $U_2(x)$ is an open set in $X$ that contains $x$ and is contained in $U_1(x)$.

Because $X$ is compact, there are finitely many elements $x_1, \ldots, x_n$ of $X$ such that

$$X = \bigcup_{l=1}^{n} U_2(x_l).$$  \hfill (47.51)

Thus

$$\sum_{l=1}^{n} \phi_{x_l}(w) > 0$$  \hfill (47.52)

for each $w \in X$, and we put

$$\psi_j(w) = \phi_{x_j}(w) \left(\sum_{l=1}^{n} \phi_{x_l}(w)\right)^{-1}$$  \hfill (47.53)

for $j = 1, \ldots, n$. This is a nonnegative real-valued continuous function on $X$ for each $j$, which satisfies $\psi_j(w) = 0$ when $w \in X \setminus U_2(x_j)$ and

$$\sum_{j=1}^{n} \psi_j(w) = 1$$  \hfill (47.54)

for every $w \in X$.

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It follows that

\[(47.55) \quad f(w, y) = \sum_{j=1}^{n} \psi_j(w)f(x_j, y) = \sum_{j=1}^{n} \psi_j(w)(f(w, y) - f(x_j, y))\]

for every \(w \in X\) and \(y \in Y\), and hence

\[(47.56) \quad |f(w, y) - \sum_{j=1}^{n} \psi_j(w)f(x_j, y)| \leq \sum_{j=1}^{n} |\psi_j(w)| |f(w, y) - f(x_j, y)|.\]

If \(\psi_j(w) \neq 0\), then \(w \in U_j(x_j) \subseteq U_i(x_j)\), so that (47.49) holds with \(x = x_j\) for every \(y \in Y\). Plugging this into (47.56), we get that

\[(47.57) \quad |f(w, y) - \sum_{j=1}^{n} \psi_j(w)f(x_j, y)| < \epsilon \sum_{j=1}^{n} \psi_j(w) = \epsilon\]

for every \(w \in X\) and \(y \in Y\), as desired.

### 47.13 Infinitely many variables

Let \(I\) be an infinite set, and suppose that \(X_i\) is a topological space for each \(i \in I\). Let \(X\) be the Cartesian product \(\prod_{i \in I} X_i\), equipped with the product topology, and let \(x_i \in X_i\) be the \(i\)th component of \(x \in X\) for each \(i \in I\). Also let \((N, \rho(u, v))\) be a metric space, and let \(f\) be a continuous mapping from \(X\) into \(N\). We would like to approximate \(f\) by functions of finitely many variables.

If \(x \in X\) and \(\epsilon > 0\), then there is an open set \(U(x)\) in \(X\) such that \(x \in U(x)\) and

\[(47.58) \quad \rho(f(x), f(y)) < \frac{\epsilon}{2}\]

for every \(y \in U(x)\). More precisely, we may as well take \(U(x)\) to be of the form

\[(47.59) \quad U(x) = \prod_{i \in I} U_i(x),\]

where \(U_i(x)\) is an open subset of \(X_i\) that contains \(x_i\) for each \(i \in I\), and

\[(47.60) \quad I(x) = \{i \in I : U_i(x) \neq X_i\}\]

is a finite subset of \(I\), because of the way that the product topology on \(X\) is defined. If \(w, y \in U(x)\), then we can apply (47.58) twice to get that

\[(47.61) \quad \rho(f(w), f(y)) \leq \rho(f(w), f(x)) + \rho(f(x), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\]

In particular, this holds when \(w \in U(x), y \in X\), and \(w_i = y_i\) for each \(i \in I(x)\), since \(y \in U(x)\) as well in this case.
Suppose now that $X_i$ is compact for each $i \in I$, so that $X$ is also compact with respect to the product topology, by Tychonoff’s theorem. This implies that there is a finite set $A \subseteq X$ such that

\[(47.62) \quad X = \bigcup_{x \in A} U(x),\]

since the $U(x)$’s form an open covering of $X$. Put

\[(47.63) \quad I_1 = \bigcup_{x \in A} I(x),\]

which is a finite subset of $I$. If $w, y \in X$ and $w_i = y_i$ for every $i \in I_1$, then (47.61) implies that

\[(47.64) \quad \rho(f(w), f(y)) < \epsilon,\]

because $w \in U(x)$ for some $x \in A$, for which $I(x) \subseteq I_1$.

\section{A Additional homework assignments}

Note that the first homework assignment was mentioned in Section 1.

\subsection{A.1 Limit points}

Let $X$ be a topological space, and for $E \subseteq X$, put

\[E' = \{ p \in X : p \text{ is a limit point of } E \}.\]

Discuss the question of whether $E'$ is closed, with examples or general results, as appropriate, e.g., how might this be related to the separation conditions?

\subsection{A.2 Dense open sets}

Let $(X, \tau)$ be a topological space. Suppose that $\tau'$ is the collection of open subsets $U$ of $X$ with respect to $\tau$ such that either $U = \emptyset$ or $U$ is dense in $X$ with respect to $\tau$. For the homework, please show the following statements. First, $\tau'$ also satisfies the requirements of a topology on $X$. Second, $X$ is automatically not Hausdorff with respect to $\tau'$ when $X$ has at least two elements. Third, suppose that $X$ satisfies the first separation condition with respect to $\tau$. If every element of $X$ is a limit point of $X$ with respect to $\tau$, then $X$ satisfies the first separation condition with respect to $\tau'$ too.

\subsection{A.3 Combining topologies}

Let $X$ be a set, and suppose that $\tau_1, \tau_2$ are topologies on $X$. Let $\tau = \tau_1 \vee \tau_2$ be the collection of subsets $W$ of $X$ such that for every $x \in W$ there are subsets $U, V$ of $X$ with $x \in U, x \in V, U \in \tau_1, V \in \tau_2$, and $U \cap V \subseteq W$. Show the following statements. First, $\tau$ is a topology on $X$. Second, $\tau_1, \tau_2 \subseteq \tau$. Third,
if \( \tau \) is any other topology on \( X \) such that \( \tau_1, \tau_2 \subseteq \tau \), then \( \tau \subseteq \tau \). Fourth, if \( d_1(x, y), d_2(x, y) \) are metrics on \( X \) for which the associated topologies are \( \tau_1, \tau_2 \), respectively, then \( d(x, y) = \max(d_1(x, y), d_2(x, y)) \) defines a metric on \( X \) for which the associated topology is \( \tau \).

### A.4 Complements of countable sets

Let \( X \) be a set. Let us say that \( U \subseteq X \) is an open set if \( U = \emptyset \) or \( X \setminus U \) has only finitely or countably many elements. Show that this defines a topology on \( X \). Of course, this is analogous to the situation mentioned previously in which the open sets are the empty set and the sets whose complements have only finitely many elements. Discuss some of the properties of this topology. For instance, under what conditions is \( X \) a Hausdorff space? Which sets \( E \subseteq X \) are dense in \( X \)? When does a sequence of elements of \( X \) converge?

### A.5 Combining topologies again

Let \( X \) be a set equipped with a topology \( \tau_1 \), and let \( \tau_2 \) be the topology on \( X \) described in Appendix A.4 in which a set \( V \subseteq X \) is an open set if \( V = \emptyset \) or \( X \setminus V \) has only finitely or countably many elements. Consider the topology \( \tau = \tau_1 \vee \tau_2 \) on \( X \) as defined in Appendix A.3. What happens when \( X \) has only finitely or countably many elements? In general, when is a point \( p \in X \) a limit point of a set \( E \subseteq X \) with respect to \( \tau \)? When is a set \( E \subseteq X \) dense in \( X \) with respect to \( \tau \)? Which sequences in \( X \) converge with respect to \( \tau \)? Does \( (X, \tau) \) satisfy the first separation condition? Is \( (X, \tau) \) Hausdorff when \( (X, \tau_1) \) is Hausdorff? If \( (X, \tau_1) \) satisfies the local countability condition at a point \( p \in X \), then under what conditions does \( (X, \tau) \) satisfy the local countability condition at \( p \)?

### A.6 Combining topologies on \( \mathbb{R} \)

Let \( \tau_1 \) be the standard topology on the real line, and let \( \tau_2 \) be the topology on \( \mathbb{R} \) defined in Appendix A.4. Thus \( U \subseteq \mathbb{R} \) is an open set with respect to \( \tau_2 \) if \( U = \emptyset \) or \( \mathbb{R} \setminus U \) has only finitely or countably many elements. Let \( \tau = \tau_1 \vee \tau_2 \) be the topology on \( \mathbb{R} \) obtained by combining these two topologies as in Appendix A.3. Show that \( (\mathbb{R}, \tau) \) is not a regular topological space, so that there is a point \( p \in \mathbb{R} \) and a closed set \( E \subseteq \mathbb{R} \) with respect to \( \tau \) such that \( p \notin E \) and \( U \cap V \neq \emptyset \) for every pair of open subsets \( U, V \) of \( \mathbb{R} \) with respect to \( \tau \) which satisfy \( p \in U \) and \( E \subseteq V \).

### A.7 Connections with product topologies

This is not so much another homework assignment, as some remarks about connections between product topologies and the way of combining topologies described in Appendix A.3. Let \( (X_1, \tau_1) \) and \( (X_2, \tau_2) \) be topological spaces. Also let \( \tau'_1 \) be the collection of subsets of \( X_1 \times X_2 \) of the form \( U_1 \times X_2 \), where \( U_1 \) is an open set in \( X_1 \) with respect to \( \tau_1 \). Similarly, let \( \tau'_2 \) be the collection of
subsets of $X_1 \times X_2$ of the form $X_1 \times U_2$, where $U_2$ is an open set in $X_2$ with respect to $\tau_2$. It is easy to see that $\tau'_1$ and $\tau'_2$ are topologies on $X_1 \times X_2$. More precisely, $\tau'_1$ is the same as the product topology on $X_1 \times X_2$ associated to $\tau_1$ on $X_1$ and the indiscrete topology on $X_2$, and $\tau'_2$ is the same as the product topology on $X_1 \times X_2$ associated to the indiscrete topology on $X_1$ and $\tau_2$ on $X_2$. The product topology on $X_1 \times X_2$ associated to $\tau_1$ on $X_1$ and $\tau_2$ on $X_2$ is the same as the combination $\tau'_1 \lor \tau'_2$ of $\tau'_1$ and $\tau'_2$ on $X_1 \times X_2$ as before. Conversely, suppose now that $\tau_1$, $\tau_2$ are topologies on the same set $X$, and let $\tau = \tau_1 \lor \tau_2$ be their combination, as before. We can also consider the product topology on $X \times X$ defined using $\tau_1$ on the first factor of $X$ and $\tau_2$ on the second factor of $X$. If we identify $X$ with the “diagonal” in $X \times X$, consisting of the ordered pairs $(x, x)$ with $x \in X$, then the combined topology $\tau = \tau_1 \lor \tau_2$ on $X$ corresponds exactly to the topology induced on the diagonal by the product topology on $X \times X$ associated to $\tau_1$ and $\tau_2$ on the two factors of $X$.

References


