Singularities of special Lagrangian submanifolds

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Abstract

We survey what is known about singularities of special Lagrangian submanifolds (SL m-folds) in (almost) Calabi–Yau manifolds. The bulk of the paper summarizes the author's work [18,19,20,21,22] on SL m-folds X with *isolated conical singularities*. That is, near each singular point x, X is modelled on an SL cone C in $\mathbb{C}^m$ with isolated singularity at 0. We also discuss directions for future research, and give a list of open problems.

1 Introduction

Special Lagrangian m-folds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$, or in *Calabi–Yau m-folds*, or more generally in *almost Calabi–Yau m-folds* (compact Kähler m-folds with trivial canonical bundle). They are of interest to Differential Geometers, to String Theorists (a species of theoretical physicist), and perhaps in the future to Algebraic Geometers.

This article will discuss the *singularities* of SL m-folds, a field which has received little attention until quite recently. We begin in §2 with a brief introduction to special Lagrangian geometry and (almost) Calabi–Yau m-folds. Sections 3–7 survey the author's series of papers [18,19,20,21,22] on SL m-folds with *isolated conical singularities*, a large class of singularities which are simple enough to study in detail. The last and longest section, §8, suggests directions for future research and gives some open problems.

We say that a compact SL m-fold X in an almost Calabi–Yau m-fold M for $m > 2$ has *isolated conical singularities* if it has only finitely many singular points $x_1, \ldots, x_n$ in M, such that for some *special Lagrangian cones* C_i in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i \setminus \{0\}$ nonsingular, X approaches C_i near x_i, in an asymptotic C^1 sense. The exact definition is given in §3.3.

Section 4 discusses the *regularity* of SL m-folds X with conical singularities $x_1, \ldots, x_n$, that is, how quickly X converges to the cone C_i near x_i, with all derivatives. In §5 we consider the *deformation theory* of compact SL m-folds X with conical singularities. We find that the *moduli space* $\mathcal{M}_X$ of deformations of X in M is locally homeomorphic to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ between finite-dimensional vector spaces, and if the *obstruction space* $\mathcal{O}_{X'}$ is zero then $\mathcal{M}_X$ is a smooth manifold.

Section 6 is an aside on *Asymptotically Conical SL m-folds (AC SL m-folds)* in $\mathbb{C}^m$, that is, nonsingular, noncompact SL m-folds L in $\mathbb{C}^m$ which are asymptotic at infinity to an SL cone C at a prescribed *rate* λ. In §7 we explain how to *desingularize* of a compact SL m-fold X with conical singularities x_i with cones C_i for $i = 1, \ldots, n$ in an almost Calabi–Yau m-fold M. We take AC SL m-folds L_i in $\mathbb{C}^m$ asymptotic to C_i at infinity, and glue tL_i into X at x_i for small $t > 0$ to get a smooth family of compact, *nonsingular* SL m-folds $\tilde{N}^t$ in M, with $\tilde{N}^t \to X$ as $t \to 0$.

For brevity I generally give only statements of results, with at most brief sketches of proofs. For the same reason I have left out several subjects I would like to discuss. Some particular omissions are:

- We give very few *examples* of SL m-folds. But many examples are known in $\mathbb{C}^m$, in [2, 4, 3, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17] and other papers.
- We give no *applications* of the results of §3–§7. See [22, §8–§10].
- We do not discuss *smooth families* of almost Calabi–Yau m-folds. However, all the main results of §2.4, §5 and §7 have extensions to families, which can be found in [18, 19, 20, 21, 22]. The discussion of *index* of singularities in §8.1, and its applications in §8.3 and §8.4, would also be improved by extending it to families.

Acknowledgements. Many people have helped me develop my ideas on special Lagrangian geometry. Amongst them I would like to thank Tom Bridgeland, Adrian Butscher, Mark Haskins, Nigel Hitchin, Stephen Marshall, Ian McIntosh, Sema Salur and Richard Thomas. I was supported by an EPSRC Advanced Research Fellowship whilst writing this paper.

2 Special Lagrangian geometry

We begin with some background from symplectic geometry. Then special Lagrangian submanifolds (SL m-folds) are introduced both in $\mathbb{C}^m$ and in *almost Calabi–Yau m-folds*. We also describe the *deformation theory* of compact SL m-folds. Some references for this section are McDuff and Salamon [26], Harvey and Lawson [3], McLean [28], and the author [13].

2.1 Background from symplectic geometry

We start by recalling some elementary symplectic geometry, which can be found in McDuff and Salamon [26]. Here are the basic definitions.

Definition 2.1. Let M be a smooth manifold of even dimension $2m$. A closed 2-form ω on M is called a *symplectic form* if the $2m$-form ω^m is nonzero at every point of M. Then (M, ω) is called a *symplectic manifold*. A submanifold N in M is called *Lagrangian* if $\dim N = m = \frac{1}{2} \dim M$ and $\omega|_N \equiv 0$.

The simplest example of a symplectic manifold is $\mathbb{R}^{2m}$.

Definition 2.2. Let $\mathbb{R}^{2m}$ have coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$, and define the standard metric g' and symplectic form ω' on $\mathbb{R}^{2m}$ by

$$g'=\sum_{j=1}^m(\mathrm{d}x_j^2+\mathrm{d}y_j^2)\quad\text{and}\quad \omega'=\sum_{j=1}^m\mathrm{d}x_j\wedge\mathrm{d}y_j.$$

Then $(\mathbb{R}^{2m},\omega')$ is a symplectic manifold. When we wish to identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$, we take the complex coordinates $(z_1,\ldots,z_m)$ on $\mathbb{C}^m$ to be $z_j=x_j+iy_j$. For $R>0$, define B_R to be the open ball of radius R about 0 in $\mathbb{R}^{2m}$.

Darboux's Theorem [26, Th. 3.15] says that every symplectic manifold is locally isomorphic to $(\mathbb{R}^{2m},\omega')$. Our version easily follows.

Theorem 2.3. *Let (M,ω) be a symplectic $2m$-manifold and $x\in M$. Then there exist $R>0$ and an embedding $\Upsilon:B_R\rightarrow M$ with $\Upsilon(0)=x$ such that $\Upsilon^*(\omega)=\omega'$, where ω' is the standard symplectic form on $\mathbb{R}^{2m}\supset B_R$. Given an isomorphism $\upsilon:\mathbb{R}^{2m}\rightarrow T_xM$ with $\upsilon^*(\omega|_x)=\omega'$, we can choose Υ with $\mathrm{d}\Upsilon|_0=\upsilon$.*

Let N be a real m-manifold. Then its tangent bundle T^*N has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1,\ldots,x_m)$ be local coordinates on N. Extend them to local coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$ on T^*N such that $(x_1,\ldots,y_m)$ represents the 1-form $y_1\mathrm{d}x_1+\cdots+y_m\mathrm{d}x_m$ in $T^*_{(x_1,\ldots,x_m)}N$. Then $\hat\omega=\mathrm{d}x_1\wedge\mathrm{d}y_1+\cdots+\mathrm{d}x_m\wedge\mathrm{d}y_m$.

Identify N with the zero section in T^*N. Then N is a *Lagrangian submanifold* of T^*N. The *Lagrangian Neighbourhood Theorem* [26, Th. 3.33] shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N.

Theorem 2.4. *Let (M,ω) be a symplectic manifold and $N\subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood U of the zero section N in T^*N, and an embedding $\Phi:U\rightarrow M$ with $\Phi|_N=\mathrm{id}:N\rightarrow N$ and $\Phi^*(\omega)=\hat\omega$, where $\hat\omega$ is the canonical symplectic structure on T^*N.*

We shall call U,Φ a *Lagrangian neighbourhood* of N. Such neighbourhoods are useful for parametrizing nearby Lagrangian submanifolds of M. Suppose that $\tilde N$ is a Lagrangian submanifold of M which is C^1-close to N. Then $\tilde N$ lies in $\Phi(U)$, and is the image $\Phi\bigl(\Gamma(\alpha)\bigr)$ of the graph $\Gamma(\alpha)$ of a unique C^1-small 1-form α on N.

As $\tilde N$ is Lagrangian and $\Phi^*(\omega)=\hat\omega$ we see that $\hat\omega|_{\Gamma(\alpha)}\equiv 0$. But one can easily show that $\hat\omega|_{\Gamma(\alpha)}=-\pi^*(\mathrm{d}\alpha)$, where $\pi:\Gamma(\alpha)\rightarrow N$ is the natural projection. Hence $\mathrm{d}\alpha=0$, and α is a *closed 1-form*. This establishes a 1-1 correspondence between small closed 1-forms on N and Lagrangian submanifolds $\tilde N$ close to N in M, which is an essential tool in proving later results.

2.2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We define *calibrations* and *calibrated submanifolds*, following [3].

Definition 2.5. Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [3, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [3, §III].

Definition 2.6. Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g', a real 2-form ω' and a complex m-form Ω' on $\mathbb{C}^m$ by

$$\begin{gathered} g' = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega' = \tfrac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega' = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{1}$$

Then g', ω' are as in Definition 2.2, and $\mathrm{Re}\,\Omega'$ and $\mathrm{Im}\,\Omega'$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\mathrm{Re}\,\Omega'$, in the sense of Definition 2.5.

Harvey and Lawson [3, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 2.7. *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega'|_L \equiv 0$ and $\mathrm{Im}\,\Omega'|_L \equiv 0$.*

Thus special Lagrangian submanifolds are *Lagrangian* submanifolds satisfying the extra condition that $\mathrm{Im}\,\Omega'|_L \equiv 0$, which is how they get their name.

2.3 Almost Calabi–Yau m-folds and SL m-folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds.*

Definition 2.8. Let $m \geqslant 2$. An *almost Calabi–Yau m-fold* is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m-dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M, and Ω is a non-vanishing holomorphic $(m, 0)$-form on M.

We call (M, J, ω, Ω) a *Calabi–Yau m-fold* if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}. \tag{2}$$

Then for each $x \in M$ there exists an isomorphism $T_xM \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g', ω', Ω' on $\mathbb{C}^m$ in (1). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

Definition 2.9. Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, and N a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$. It easily follows that $\operatorname{Re}\Omega|_N$ is a nonvanishing m-form on N. Thus N is orientable, with a unique orientation in which $\operatorname{Re}\Omega|_N$ is positive.

Again, this is not the usual definition of SL m-fold, but is essentially equivalent to it. Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold, with metric g. Let $\psi : M \to (0, \infty)$ be the unique smooth function such that

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}, \tag{3}$$

and define $\tilde{g}$ to be the conformally equivalent metric $\psi^2 g$ on M. Then $\operatorname{Re}\Omega$ is a *calibration* on the Riemannian manifold $(M, \tilde{g})$, and SL m-folds N in (M, J, ω, Ω) are calibrated with respect to it, so that they are minimal with respect to $\tilde{g}$.

If M is a Calabi–Yau m-fold then $\psi \equiv 1$ by (2), so $\tilde{g} = g$, and an m-submanifold N in M is special Lagrangian if and only if it is calibrated w.r.t. $\operatorname{Re}\Omega$ on (M, g), as in Definition 2.6. This recovers the usual definition of special Lagrangian m-folds in Calabi–Yau m-folds.

2.4 Deformations of compact SL m-folds

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [28, §3], who proved the following result in the Calabi–Yau case. The extension to the almost Calabi–Yau case is described in [13, §9.5].

Theorem 2.10. *Let N be a compact SL m-fold in an almost Calabi–Yau m-fold (M, J, ω, Ω). Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$.*

Sketch of proof. There is a natural orthogonal decomposition $TM|_N = TN \oplus \nu$, where $\nu \to N$ is the *normal bundle* of N in M. As N is Lagrangian, the complex structure $J : TM \to TM$ gives an isomorphism $J : \nu \to TN$. But the metric g gives an isomorphism $TN \cong T^*N$. Composing these two gives an isomorphism $\nu \cong T^*N$.

Let T be a small *tubular neighborhood* of N in M. Then we can identify T with a neighborhood of the zero section in ν. Using the isomorphism $\nu \cong T^*N$, we have an identification between T and a neighborhood of the zero section in T^*N. This can be chosen to identify the Kähler form ω on T with the natural symplectic structure on T^*N. Let $\pi : T \to N$ be the obvious projection.

Under this identification, submanifolds N' in $T \subset M$ which are C^1 close to N are identified with the graphs of small smooth sections α of T^*N. That is, submanifolds N' of M close to N are identified with 1-*forms* α on N. We need to know: which 1-forms α are identified with *special Lagrangian* submanifolds N'?

Well, N' is special Lagrangian if $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$. Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so we can push $\omega|_{N'}$ and $\operatorname{Im}\Omega|_{N'}$ down to N, and regard them as functions of α. Calculation shows that $\pi_*\bigl(\omega|_{N'}\bigr) = \mathrm{d}\alpha$ and $\pi_*\bigl(\operatorname{Im}\Omega|_{N'}\bigr) = F(\alpha, \nabla\alpha)$, where F is a nonlinear function of its arguments. Thus, the moduli space $\mathcal{M}_{N}$ is locally isomorphic to the set of small 1-forms α on N such that $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$.

Now it turns out that F satisfies $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ when α is small. Therefore $\mathcal{M}_{N}$ is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N, \mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of T^*N, Λ^2T^*N and Λ^mT^*N, use *elliptic regularity results* to show the map $\alpha \mapsto \bigl(\mathrm{d}\alpha, F(\alpha, \nabla\alpha)\bigr)$ has *closed image* in these Banach spaces, and then use the *Implicit Function Theorem for Banach spaces* to show that the kernel of the map is what we expect. □

3 SL cones and conical singularities

We begin in §3.1 with some definitions on *special Lagrangian cones*. Section 3.2 gives *examples* of SL cones, and §3.3 defines *SL m-folds with conical singularities*, the subject of the paper. Section 3.4 discusses *homology* and *cohomology* of SL m-folds with conical singularities.

3.1 Preliminaries on special Lagrangian cones

We define *special Lagrangian cones*, and some notation.

Definition 3.1. A (singular) SL m-fold C in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{t\,\mathbf{x} : \mathbf{x} \in C\}$. Let C be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m–1)$-submanifold of $\mathcal{S}^{2m-1}$, not necessarily connected. Let g_{Σ} be the restriction of g' to Σ, where g' is as in (1).

Set $C' = C \setminus \{0\}$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then ι has image C'. By an abuse of notation, *identify* C' with $\Sigma \times (0, \infty)$ using ι. The *cone metric* on $C' \cong \Sigma \times (0, \infty)$ is $g' = \iota^*(g') = \mathrm{d}r^2 + r^2 g_{\Sigma}$.

For $\alpha \in \mathbb{R}$, we say that a function $u : C' \to \mathbb{R}$ is *homogeneous of order* α if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, u is homogeneous of order α if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \to \mathbb{R}$.

In [18, Lem. 2.3] we study *homogeneous harmonic functions* on C'.

Lemma 3.2. *In the situation of Definition 3.1, let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order α on $C' = \Sigma \times (0, \infty)$, for $v \in C^2(\Sigma)$. Then*

$$\Delta u(\sigma, r) = r^{\alpha-2}\bigl(\Delta_{\Sigma} v - \alpha(\alpha + m - 2)v\bigr),$$

where Δ, Δ_Σ are the Laplacians on (C', g') and (Σ, g_Σ). Hence, u is harmonic on C' if and only if v is an eigenfunction of Δ_Σ with eigenvalue $\alpha(\alpha+m-2)$.

Following [18, Def. 2.5], we define:

Definition 3.3. In Definition 3.1, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \bigl\{\alpha \in \mathbb{R} : \alpha(\alpha+m-2) \text{ is an eigenvalue of } \Delta_\Sigma\bigr\}. \tag{4}$$

Then $\mathcal{D}_\Sigma$ is a countable, discrete subset of $\mathbb{R}$. By Lemma 3.2, an equivalent definition is that $\mathcal{D}_\Sigma$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function u of order α on C'.

Define $m_\Sigma : \mathcal{D}_\Sigma \to \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha+m-2)$ of Δ_Σ, or equivalently the dimension of the vector space of homogeneous harmonic functions u of order α on C'. Define $N_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$N_\Sigma(\delta) = -\sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta,0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \text{ and } N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0,\delta]} m_\Sigma(\alpha) \text{ if } \delta \geqslant 0. \tag{5}$$

Then N_Σ is monotone increasing and upper semicontinuous, and is discontinuous exactly on $\mathcal{D}_\Sigma$, increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of Δ_Σ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2-m, 0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2-m, 0)$.

We define the *stability index* of C, and *stable* and *rigid* cones [19, Def. 3.6].

Definition 3.4. Let C be an SL cone in $\mathbb{C}^m$ for $m > 2$ with an isolated singularity at 0, let G be the Lie subgroup of $\mathrm{SU}(m)$ preserving C, and use the notation of Definitions 3.1 and 3.3. Then [19, eq. (8)] shows that

$$m_\Sigma(0) = b^0(\Sigma), \quad m_\Sigma(1) \geqslant 2m \quad \text{and} \quad m_\Sigma(2) \geqslant m^2 - 1 - \dim G. \tag{6}$$

Define the *stability index* s-ind(C) to be

$$\text{s-ind}(C) = N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G.$$

Then s-ind$(C) \geqslant 0$ by (6), as $N_\Sigma(2) \geqslant m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$ by (5). We call C *stable* if s-ind$(C) = 0$.

Following [18, Def. 6.7], we call C *rigid* if $m_\Sigma(2) = m^2 - 1 - \dim G$. As

$$\text{s-ind}(C) \geqslant m_\Sigma(2) - (m^2 - 1 - \dim G) \geqslant 0,$$

we see that *if C is stable, then C is rigid*.

We shall see in §5 that s-ind(C) is the dimension of an obstruction space to deforming an SL m-fold X with a conical singularity with cone C, and that if C is *stable* then the deformation theory of X simplifies. An SL cone C is *rigid* if all infinitesimal deformations of C as an SL cone come from $\mathrm{SU}(m)$ rotations of C. This will be useful in the Geometric Measure Theory material of §4.

3.2 Examples of special Lagrangian cones

In our first example we can compute the data of §3.1 very explicitly.

Example 3.5. Here is a family of special Lagrangian cones constructed by Harvey and Lawson [3, §III.3.A]. For $m \geqslant 3$, define

$$C_{\mathrm{HL}}^m = \bigl\{(z_1,\ldots,z_m)\in\mathbb{C}^m : i^{m+1}z_1\cdots z_m\in[0,\infty), \quad |z_1|=\cdots=|z_m|\bigr\}. \quad (7)$$

Then C_{HL}^m is a *special Lagrangian cone* in $\mathbb{C}^m$ with an isolated singularity at 0, and $\Sigma_{\mathrm{HL}}^m = C_{\mathrm{HL}}^m\cap\mathcal{S}^{2m-1}$ is an $(m-1)$-torus T^{m-1}. Both C_{HL}^m and Σ_{HL}^m are invariant under the $\mathrm{U}(1)^{m-1}$ subgroup of $\mathrm{SU}(m)$ acting by

$$(z_1,\ldots,z_m)\mapsto(\mathrm{e}^{i\theta_1}z_1,\ldots,\mathrm{e}^{i\theta_m}z_m) \quad \text{for } \theta_j\in\mathbb{R} \text{ with } \theta_1+\cdots+\theta_m=0. \quad (8)$$

In fact $\pm C_{\mathrm{HL}}^m$ for m odd, and $C_{\mathrm{HL}}^m, iC_{\mathrm{HL}}^m$ for m even, are the unique SL cones in $\mathbb{C}^m$ invariant under (8), which is how Harvey and Lawson constructed them.

The metric on $\Sigma_{\mathrm{HL}}^m\cong T^{m-1}$ is flat, so it is easy to compute the eigenvalues of $\Delta_{\Sigma_{\mathrm{HL}}^m}$. This was done by Marshall [25, §6.3.4]. There is a 1-1 correspondence between $(n_1,\ldots,n_{m-1})\in\mathbb{Z}^{m-1}$ and eigenvectors of $\Delta_{\Sigma_{\mathrm{HL}}^m}$ with eigenvalue

$$m\sum_{i=1}^{m-1}n_i^2-\sum_{i,j=1}^{m-1}n_in_j. \quad (9)$$

Using (9) and a computer we can find the eigenvalues of $\Delta_{\Sigma_{\mathrm{HL}}^m}$ and their multiplicities. The Lie subgroup G_{HL}^m of $\mathrm{SU}(m)$ preserving C_{HL}^m has identity component the $\mathrm{U}(1)^{m-1}$ of (8), so that $\dim G_{\mathrm{HL}}^m = m-1$. Thus we can calculate $\mathcal{D}_{\Sigma_{\mathrm{HL}}^m}$, $m_{\Sigma_{\mathrm{HL}}^m}$, $N_{\Sigma_{\mathrm{HL}}^m}$, and the stability index s-ind(C_{HL}^m). This was done by Marshall [25, Table 6.1] and the author [19, §3.2]. Table 1 gives the data $m, N_{\Sigma_{\mathrm{HL}}^m}(2), m_{\Sigma_{\mathrm{HL}}^m}(2)$ and s-ind(C_{HL}^m) for $3\leqslant m\leqslant 12$.

m	3	4	5	6	7	8	9	10	11	12
$N_{\Sigma_{\mathrm{HL}}^m}(2)$	13	27	51	93	169	311	331	201	243	289
$m_{\Sigma_{\mathrm{HL}}^m}(2)$	6	12	20	30	42	126	240	90	110	132
s-ind(C_{HL}^m)	0	6	20	50	112	238	240	90	110	132

Table 1: Data for $\mathrm{U}(1)^{m-1}$-invariant SL cones C_{HL}^m in $\mathbb{C}^m$

One can also prove that

$$N_{\Sigma_{\mathrm{HL}}^m}(2)=2m^2+1 \text{ and } m_{\Sigma_{\mathrm{HL}}^m}(2)=\text{s-ind}(C_{\mathrm{HL}}^m)=m^2-m \text{ for } m\geqslant 10. \quad (10)$$

As C_{HL}^m is *stable* when s-ind$(C_{\mathrm{HL}}^m)=0$ we see from Table 1 and (10) that C_{HL}^3 is a *stable* cone in $\mathbb{C}^3$, but C_{HL}^m is *unstable* for $m\geqslant 4$. Also C_{HL}^m is *rigid* when $m_{\Sigma_{\mathrm{HL}}^m}(2)=m^2-m$, as $\dim G_{\mathrm{HL}}^m=m-1$. Thus C_{HL}^m is *rigid* if and only if $m\neq 8,9$, by Table 1 and (10).

Here is an example chosen from [7, Ex. 9.4] as it is easy to write down.

Example 3.6. Let $a_1, \ldots, a_m \in \mathbb{Z}$ with $a_1 + \cdots + a_m = 0$ and highest common factor 1, such that $a_1, \ldots, a_k > 0$ and $a_{k+1}, \ldots, a_m < 0$ for $0 < k < m$. Define

$$L_0^{a_1,\ldots,a_m} = \bigl\{ \bigl(i\mathrm{e}^{ia_1\theta} x_1, \mathrm{e}^{ia_2\theta} x_2, \ldots, \mathrm{e}^{ia_m\theta} x_m \bigr) : \theta \in [0, 2\pi), \\ x_1, \ldots, x_m \in \mathbb{R}, \qquad a_1 x_1^2 + \cdots + a_m x_m^2 = 0 \bigr\}.$$

Then $L_0^{a_1,\ldots,a_m}$ is an *immersed SL cone* in $\mathbb{C}^m$, with an isolated singularity at 0.

Define $C^{a_1,\ldots,a_m} = \bigl\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : a_1 x_1^2 + \cdots + a_m x_m^2 = 0 \bigr\}$. Then $C^{a_1,\ldots,a_m}$ is a quadric cone on $\mathcal{S}^{k-1} \times S^{m-k-1}$ in $\mathbb{R}^m$, and $L_0^{a_1,\ldots,a_m}$ is the image of an immersion $\Phi : C^{a_1,\ldots,a_m} \times \mathcal{S}^1 \to \mathbb{C}^m$, which is generically 2:1. Therefore $L_0^{a_1,\ldots,a_m}$ is an immersed SL cone on $(\mathcal{S}^{k-1} \times \mathcal{S}^{m-k-1} \times \mathcal{S}^1)/\mathbb{Z}_2$.

Further examples of SL cones are constructed by Harvey and Lawson [3, §III.3], Haskins [4], the author [7, 8], and others. Special Lagrangian cones in $\mathbb{C}^3$ are a special case, which may be treated using the theory of *integrable systems*. In principle this should yield a *classification* of all SL cones on T^2 in $\mathbb{C}^3$. For more information see McIntosh [27] or the author [12].

3.3 Special Lagrangian m-folds with conical singularities

Now we can define *conical singularities* of SL m-folds, following [18, Def. 3.6].

Definition 3.7. Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold for $m > 2$, and define $\psi : M \to (0, \infty)$ as in (3). Suppose X is a compact singular SL m-fold in M with singularities at distinct points $x_1, \ldots, x_n \in X$, and no other singularities.

Fix isomorphisms $\upsilon_i : \mathbb{C}^m \to T_{x_i} M$ for $i = 1, \ldots, n$ such that $\upsilon_i^*(\omega) = \omega'$ and $\upsilon_i^*(\Omega) = \psi(x_i)^m \Omega'$, where ω', Ω' are as in (1). Let $C_1, \ldots, C_n$ be SL cones in $\mathbb{C}^m$ with isolated singularities at 0. For $i = 1, \ldots, n$ let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and let $\mu_i \in (2, 3)$ with

$$(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset, \quad \text{where } \mathcal{D}_{\Sigma_i} \text{ is defined in (4).} \tag{11}$$

Then we say that X has a *conical singularity* or *conical singular point* at x_i, with *rate* μ_i and *cone* C_i for $i = 1, \ldots, n$, if the following holds.

By Theorem 2.3 there exist embeddings $\Upsilon_i : B_R \to M$ for $i = 1, \ldots, n$ satisfying $\Upsilon_i(0) = x_i$, $\mathrm{d}\Upsilon_i|_0 = \upsilon_i$ and $\Upsilon_i^*(\omega) = \omega'$, where B_R is the open ball of radius R about 0 in $\mathbb{C}^m$ for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \to B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \ldots, n$.

Define $X' = X \setminus \{x_1, \ldots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets $S_1, \ldots, S_n$ with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \ldots, \bar{S}_n$ are disjoint in X. For $i = 1, \ldots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \to B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$, and

$$\bigl| \nabla^k (\phi_i - \iota_i) \bigr| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1. \tag{12}$$

Here ∇ is the Levi-Civita connection of the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$, $|\,.\,|$ is computed using $\iota_i^*(g')$. If the cones $C_1, \ldots, C_n$ are *stable* in the sense of Definition 3.4, then we say that X has *stable conical singularities.*

We will see in Theorems 4.1 and 4.2 that if (12) holds for $k = 0, 1$ and some μ_i satisfying (11), then we can choose a natural ϕ_i for which (12) holds for *all* $k \geqslant 0$, and for *all* rates μ_i satisfying (11). Thus the number of derivatives required in (12) and the choice of μ_i both make little difference. We choose $k = 0, 1$ in (12), and some μ_i in (11), to make the definition as weak as possible.

We suppose $m > 2$ for two reasons. Firstly, the only SL cones in $\mathbb{C}^2$ are finite unions of SL planes $\mathbb{R}^2$ in $\mathbb{C}^2$ intersecting only at 0. Thus any SL 2-fold with conical singularities is actually *nonsingular* as an immersed 2-fold, so there is really no point in studying them. Secondly, $m = 2$ is a special case in the analysis of [18, §2], and it is simpler to exclude it. Therefore we will suppose $m > 2$ throughout the paper.

Here are the reasons for the conditions on μ_i in Definition 3.7:

- We need $\mu_i > 2$, or else (12) does not force X to approach C_i near x_i.
- The definition involves a choice of $\Upsilon_i : B_R \to M$. If we replace Υ_i by a different choice $\tilde{\Upsilon}_i$ then we should replace ϕ_i by $\tilde{\phi}_i = (\tilde{\Upsilon}_i^{-1} \circ \Upsilon_i) \circ \phi_i$ near 0 in B_R. Calculation shows that as $\Upsilon_i, \tilde{\Upsilon}_i$ agree up to second order, we have $\left|\nabla^k(\tilde{\phi}_i - \phi_i)\right| = O(r^{2-k})$.

  Therefore we choose $\mu_i < 3$ so that these $O(r^{2-k})$ terms are absorbed into the $O(r^{\mu_i - 1 - k})$ in (12). This makes the definition independent of the choice of Υ_i, which it would not be if $\mu_i > 3$.
- Condition (11) is needed to prove the regularity result Theorem 4.2, and also to reduce to a minimum the obstructions to deforming compact SL m-folds with conical singularities studied in §5.

3.4 Homology and cohomology

Next we discuss *homology* and *cohomology* of SL m-folds with conical singularities, following [18, §2.4]. For a general reference, see for instance Bredon [1]. When Y is a manifold, write $H^k(Y, \mathbb{R})$ for the k^{th} *de Rham cohomology group* and $H^k_{\text{cs}}(Y, \mathbb{R})$ for the k^{th} *compactly-supported de Rham cohomology group* of Y. If Y is compact then $H^k(Y, \mathbb{R}) = H^k_{\text{cs}}(Y, \mathbb{R})$. The *Betti numbers* of Y are $b^k(Y) = \dim H^k(Y, \mathbb{R})$ and $b^k_{\text{cs}}(Y) = \dim H^k_{\text{cs}}(Y, \mathbb{R})$.

Let Y be a topological space, and $Z \subset Y$ a subspace. Write $H_k(Y, \mathbb{R})$ for the k^{th} *real singular homology group* of Y, and $H_k(Y; Z, \mathbb{R})$ for the k^{th} *real singular relative homology group* of $(Y; Z)$. When Y is a manifold and Z a submanifold we define $H_k(Y, \mathbb{R})$ and $H_k(Y; Z, \mathbb{R})$ using *smooth* simplices, as in [1, §V.5]. Then the pairing between (singular) homology and (de Rham) cohomology is defined at the chain level by integrating k-forms over k-simplices.

Let X be a compact SL m-fold in M with conical singularities $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$, and set $X' = X \setminus \{x_1, \ldots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, as in

§3.3. Then X' is the interior of a compact manifold $\bar{X}'$ with boundary $\coprod_{i=1}^n \Sigma_i$. Using this we show in [18, §2.4] that there is a natural long exact sequence

$$\cdots \to H^k_{\rm cs}(X',\mathbb{R}) \to H^k(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^k(\Sigma_i,\mathbb{R}) \to H^{k+1}_{\rm cs}(X',\mathbb{R}) \to \cdots , \tag{13}$$

and natural isomorphisms

$$\begin{gathered} H_k\bigl(X;\{x_1,\ldots,x_n\},\mathbb{R}\bigr)^* \cong H^k_{\rm cs}(X',\mathbb{R}) \cong H_{m-k}(X',\mathbb{R}) \cong H^{m-k}(X',\mathbb{R})^* \\ \text{and} \quad H^k_{\rm cs}(X',\mathbb{R}) \cong H_k(X,\mathbb{R})^* \quad \text{for all } k>1. \end{gathered}$$

The inclusion $\iota : X \to M$ induces homomorphisms $\iota_* : H_k(X,\mathbb{R}) \to H_k(M,\mathbb{R})$.

4 The asymptotic behaviour of X near x_i

We now review the work of [18] on the *regularity* of SL m-folds with conical singularities. Let M be an almost Calabi–Yau m-fold and X an SL m-fold in M with conical singularities at $x_1,\ldots,x_n$, with identifications υ_i and cones C_i. We study how quickly X converges to the cone $\upsilon(C_i)$ in $T_{x_i}M$ near x_i.

Roughly speaking, we work by arranging for ϕ_i in Definition 3.7 to satisfy an *elliptic equation*, and then use *elliptic regularity results* to deduce asymptotic bounds for $\phi_i - \iota_i$ and all its derivatives. Now ϕ_i is not uniquely defined, but is a more-or-less arbitrary parametrization of $\Upsilon_i^*(X')$ near 0 in $\mathbb{C}^m$. To make ϕ_i satisfy an elliptic equation we impose an *extra condition*, that $(\phi_i - \iota_i)(\sigma,r)$ is orthogonal to $T_{\iota_i(\sigma,r)}C_i$ w.r.t. the metric g' on $\mathbb{C}^m$, for all $(\sigma,r)\in\Sigma_i\times(0,R')$. By [18, Th. 4.4] this also fixes ϕ_i uniquely, given $\upsilon_i, R, \Upsilon_i$ and R'.

Theorem 4.1. *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ with identifications $\upsilon_i:\mathbb{C}^m\to T_{x_i}M$ and cones $C_1,\ldots,C_n$. Choose $R>0$ and $\Upsilon_i : B_R\to M$ as in Definition 3.7. Then for sufficiently small $R'\in(0,R]$ there exist unique ϕ_i, S_i for $i=1,\ldots,n$ satisfying the conditions of Definition 3.7 and*

$$(\phi_i-\iota_i)(\sigma,r)\perp T_{\iota_i(\sigma,r)}C_i \quad \text{in } \mathbb{C}^m \text{ for all } (\sigma,r)\in\Sigma_i\times(0,R'). \tag{14}$$

In fact [18, Th. 4.4] characterizes ϕ_i in terms of a *Lagrangian neighbourhood* U_{C_i}, Φ_{C_i} of C_i in $\mathbb{C}^m$, but examining the proof of [18, Th. 4.2] shows this is equivalent to (14). In [18, §5] we study the asymptotic behaviour of the maps ϕ_i of Theorem 4.1. Combining [18, Th.s 5.1 & 5.5, Lem. 4.5] proves:

Theorem 4.2. *In the situation of Theorem 4.1, suppose $\mu_i'\in(2,3)$ with $(2,\mu_i']\cap\mathcal{D}_{\Sigma_i}=\emptyset$ for $i=1,\ldots,n$. Then*

$$\bigl|\nabla^k(\phi_i-\iota_i)\bigr| = O(r^{\mu_i'-1-k}) \qquad \text{for all } k\geqslant 0 \text{ and } i=1,\ldots,n. \tag{15}$$

Hence X has conical singularities at x_i with cone C_i and rate μ_i', for all possible rates μ_i' allowed by Definition 3.7. Therefore, the definition of conical singularities is essentially independent of the choice of rate μ_i.

Theorem 4.2 in effect *strengthens* the definition of SL m-folds with conical singularities, Definition 3.7, as it shows that (12) actually implies the much stronger condition (15) on all derivatives.

To prove Theorem 4.2, we show using an analogue of Theorem 2.4 for C_i in $\mathbb{C}^m$ that as $\Upsilon_i^*(X')$ is *Lagrangian* in B_R, we may regard ϕ_i as the graph of a *closed* 1-*form* η_i on $\Sigma_i \times (0, R')$. The asymptotic condition (12) implies that η_i is *exact*, so we may write $\eta_i = \mathrm{d}A_i$ for smooth $A_i : \Sigma_i \times (0, R') \to \mathbb{R}$. As $\operatorname{Im} \Omega|_{X'} \equiv 0$, we find that A_i satisfies the second-order nonlinear p.d.e.

$$\mathrm{d}^*\bigl(\psi^m \mathrm{d}A_i\bigr)(\sigma, r) = Q\bigl(\sigma, r, \mathrm{d}A_i(\sigma, r), \nabla^2 A_i(\sigma, r)\bigr) \tag{16}$$

for $(\sigma, r) \in \Sigma_i \times (0, R')$, where Q is a smooth nonlinear function.

When r is small the Q term in (16) is also small and (16) approximates $\Delta_i A_i = 0$, where Δ_i is the Laplacian on the cone C_i. Therefore (16) is *elliptic* for small r. Using known results on the regularity of solutions of nonlinear second-order elliptic p.d.e.s, and a theory of analysis on weighted Sobolev spaces on manifolds with cylindrical ends developed by Lockhart and McOwen [24], we can then prove (15).

Our next result [18, Th. 6.8] is an application of *Geometric Measure Theory*. For an introduction to the subject, see Morgan [29]. Geometric Measure Theory studies measure-theoretic generalizations of submanifolds called *integral currents*, which may be very singular, and is particularly powerful for *minimal* submanifolds. As from §2 SL m-folds are minimal submanifolds w.r.t. an appropriate metric, many major results of Geometric Measure Theory immediately apply to *special Lagrangian integral currents*, a very general class of singular SL m-folds with strong compactness properties.

Theorem 4.3. *Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and define $\psi : M \to (0, \infty)$ as in* (3)*. Let $x \in M$ and fix an isomorphism $\upsilon : \mathbb{C}^m \to T_x M$ with $\upsilon^*(\omega) = \omega'$ and $\upsilon^*(\Omega) = \psi(x)^m \Omega'$, where ω', Ω' are as in* (1)*.*

Suppose that T is a special Lagrangian integral current in M with $x \in T^\circ$, where $T^\circ = \operatorname{supp} T \setminus \operatorname{supp} \partial T$, and that $\upsilon_*(C)$ is a multiplicity 1 *tangent cone to T at x, where C is a rigid special Lagrangian cone in $\mathbb{C}^m$, in the sense of Definition 3.4. Then T has a conical singularity at x, in the sense of Definition 3.7.*

This is a *weakening* of Definition 3.7 for *rigid* cones C. Theorem 4.3 also holds for the larger class of *Jacobi integrable* SL cones C, defined in [18, Def. 6.7]. Basically, Theorem 4.3 shows that if a singular SL m-fold T in M is locally modelled on a rigid SL cone C in only a very weak sense, then it necessarily satisfies Definition 3.7. One moral of Theorems 4.2 and 4.3 is that, at least for rigid SL cones C, more-or-less *any* sensible definition of SL m-folds with conical singularities is equivalent to Definition 3.7.

Theorem 4.3 is proved by applying regularity results of Allard and Almgren, and Adams and Simon, mildly adapted to the special Lagrangian situation, which roughly say that if a tangent cone C_i to X at x_i has an isolated singularity at 0, is multiplicity 1, and rigid, then X has a parametrization ϕ_i near x_i which satisfies (12) for some $\mu_i > 2$. It then quickly follows that X has a conical singularity at x_i, in the sense of Definition 3.7.

As discussed in [18, §6.3], one can use other results from Geometric Measure Theory to argue that for tangent cones C which are not Jacobi integrable, Definition 3.7 may be *too strong*, in that there could exist examples of singular SL m-folds with tangent cone C which are not covered by Definition 3.7, as the decay conditions (12) are too strict.

5 Moduli of SL m-folds with conical singularities

Next we review the work of [19] on *deformation theory* for compact SL m-folds with conical singularities. Following [19, Def. 5.4], we define the space $\mathcal{M}_X$ of compact SL m-folds $\hat{X}$ in M with conical singularities deforming a fixed SL m-fold X with conical singularities.

Definition 5.1. Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ and cones $C_1, \ldots, C_n$. Define the *moduli space* $\mathcal{M}_X$ *of deformations of* X to be the set of $\hat{X}$ such that

(i) $\hat{X}$ is a compact SL m-fold in M with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$ with cones $C_1, \ldots, C_n$, for some $\hat{x}_i$ and identifications $\hat{\upsilon}_i : \mathbb{C}^m \to T_{\hat{x}_i}M$.

(ii) There exists a homeomorphism $\hat{\iota} : X \to \hat{X}$ with $\hat{\iota}(x_i) = \hat{x}_i$ for $i = 1, \ldots, n$ such that $\hat{\iota}|_{X'} : X' \to \hat{X}'$ is a diffeomorphism and $\hat{\iota}$ and ι are isotopic as continuous maps $X \to M$, where $\iota : X \to M$ is the inclusion.

In [19, Def. 5.6] we define a *topology* on $\mathcal{M}_X$, and explain why it is the natural choice. We will not repeat the complicated definition here; readers are referred to [19, §5] for details. In [19, Th. 6.10] we describe $\mathcal{M}_X$ near X, in terms of a smooth map Φ between the *infinitesimal deformation space* $\mathcal{I}_{X'}$ and the *obstruction space* $\mathcal{O}_{X'}$.

Theorem 5.2. *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $\mathcal{M}_X$ be the moduli space of deformations of X as an SL m-fold with conical singularities in M, as in Definition 5.1. Set $X' = X \setminus \{x_1, \ldots, x_n\}$.*

Then there exist natural finite-dimensional vector spaces $\mathcal{I}_{X'}$, $\mathcal{O}_{X'}$ such that $\mathcal{I}_{X'}$ is isomorphic to the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$ and $\dim \mathcal{O}_{X'} = \sum_{i=1}^n \text{s-ind}(C_i)$, where $\text{s-ind}(C_i)$ is the stability index of Definition 3.4. There exists an open neighbourhood U of 0 in $\mathcal{I}_{X'}$, a smooth map $\Phi : U \to \mathcal{O}_{X'}$ with $\Phi(0) = 0$, and a map $\Xi : \{u \in U : \Phi(u) = 0\} \to \mathcal{M}_X$ with $\Xi(0) = X$ which is a homeomorphism with an open neighbourhood of X in $\mathcal{M}_X$.

Here is a sketch of the proof. For simplicity, consider first the subset of $\hat{X} \in \mathcal{M}_X$ which have the same singular points $x_1, \ldots, x_n$ and identifications $\upsilon_1, \ldots, \upsilon_n$ as X. Generalizing Theorem 2.10, in [18, Th. 4.3] we define a *Lagrangian neighbourhood* $U_{X'}, \Phi_{X'}$ for X', with certain compatibilities with Υ_i, ϕ_i

near x_i. If $\hat{X}$ is C^1 close to X in an appropriate sense then $\hat{X}' = \Phi_{X'}\big(\Gamma(\alpha)\big)$, where $\Gamma(\alpha) \subset U_{X'}$ is the graph of a small 1-form α on X'.

Since $\hat{X}'$ is Lagrangian, α is *closed*, as in §2.1. Also, applying Theorem 4.2 to $X, \hat{X}$ and noting that α on S_i corresponds to $\hat{\phi}_i - \phi_i$ on $\Sigma_i \times (0, R')$, we find that if $i = 1, \ldots, n$ and $\mu_i' \in (2, 3)$ with $(2, \mu_i'] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ then

$$\big|\nabla^k \alpha(x)\big| = O\big(d(x, x_i)^{\mu_i' - 1 - k}\big) \quad \text{near } x_i \text{ for all } k \geqslant 0. \tag{17}$$

As α is closed it has a cohomology class $[\alpha] \in H^1(X', \mathbb{R})$. Since (17) implies that α decays quickly near x_i, it turns out that α must be *exact* near x_i. Therefore $[\alpha]$ can be represented by a compactly-supported form on X', and lies in the image of $H^1_{\rm cs}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$.

Choose a vector space $\mathcal{I}_{X'}$ of compactly-supported 1-forms on X' representing the image of $H^1_{\rm cs}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$. Then we can write $\alpha = \beta + {\rm d}f$, where $\beta \in \mathcal{I}_{X'}$ with $[\alpha] = [\beta]$ is unique, and $f \in C^\infty(X')$ is unique up to addition of constants. As $\hat{X}'$ is special Lagrangian we find that f satisfies a *second-order nonlinear elliptic p.d.e.* similar to (16):

$${\rm d}^*\big(\psi^m(\beta + {\rm d}f)\big)(x) = Q\big(x, (\beta + {\rm d}f)(x), (\nabla\beta + \nabla^2 f)(x)\big) \tag{18}$$

for $x \in X'$. The *linearization* of (18) at $\beta = f = 0$ is ${\rm d}^*\big(\psi^m(\beta + {\rm d}f)\big) = 0$.

We study the family of small solutions β, f of (18) for which f has the decay near x_i required by (17). There is a ready-made theory of analysis on manifolds with cylindrical ends developed by Lockhart and McOwen [24], which is well-suited to this task. We work on certain *weighted Sobolev spaces* $L^p_{k,\boldsymbol{\mu}}(X')$ of functions on X'.

By results from [24] it turns out that the operator $f \mapsto {\rm d}^*(\psi^m {\rm d}f)$ is a *Fredholm* map $L^p_{k,\boldsymbol{\mu}}(X') \to L^p_{k-2,\boldsymbol{\mu}-2}(X')$, with cokernel of dimension $\sum_{i=1}^n N_{\Sigma_i}(2)$. This cokernel is in effect the *obstruction space* to deforming X with x_i, υ_i fixed, as it is the obstruction space to solving the linearization of (18) in f at $\beta = f = 0$.

By varying the x_i and υ_i, and allowing f to converge to different constant values on the ends of X' rather than zero, we can overcome many of these obstructions. This reduces the dimension of the obstruction space $\mathcal{O}_{X'}$ from $\sum_{i=1}^n N_{\Sigma_i}(2)$ to $\sum_{i=1}^n \text{s-ind}(C_i)$. The obstruction map Φ is constructed using the Implicit Mapping Theorem for Banach spaces. This concludes our sketch.

If the C_i are *stable* then $\mathcal{O}_{X'} = \{0\}$ and we deduce [19, Cor. 6.11]:

Corollary 5.3. *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with stable conical singularities, and let $\mathcal{M}_X$ and $\mathcal{I}_{X'}$ be as in Theorem 5.2. Then $\mathcal{M}_X$ is a smooth manifold of dimension* $\dim \mathcal{I}_{X'}$.

This has clear similarities with Theorem 2.10. Here is another simple condition for $\mathcal{M}_X$ to be a manifold near X, [19, Def. 6.12].

Definition 5.4. Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities, and let $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U$ and Φ be as in Theorem 5.2. We call X *transverse* if the linear map ${\rm d}\Phi|_0 : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ is surjective.

If X is transverse then $\{u \in U : \Phi(u) = 0\}$ is a manifold near 0, so Theorem 5.2 yields [19, Cor. 6.13]:

Corollary 5.5. *Suppose* (M, J, ω, Ω) *is an almost Calabi–Yau m-fold and X a transverse compact SL m-fold in M with conical singularities, and let $\mathcal{M}_X, \mathcal{I}_{X'}$ and $\mathcal{O}_{X'}$ be as in Theorem 5.2. Then $\mathcal{M}_X$ is near X a smooth manifold of dimension* $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$.

Now there are a number of well-known moduli space problems in geometry where in general moduli spaces are obstructed and singular, but after a generic perturbation they become smooth manifolds. For instance, moduli spaces of instantons on 4-manifolds can be made smooth by choosing a generic metric, and similar things hold for Seiberg–Witten equations, and moduli spaces of pseudo-holomorphic curves in symplectic manifolds.

In [19, §9] we try (but do not quite succeed) to replicate this for moduli spaces of SL m-folds with conical singularities, by choosing a *generic Kähler metric* in a fixed Kähler class. This is the idea behind [19, Conj. 9.5]:

Conjecture 5.6. *Let* (M, J, ω, Ω) *be an almost Calabi–Yau m-fold, X a compact SL m-fold in M with conical singularities, and $\mathcal{I}_{X'}, \mathcal{O}_{X'}$ be as in Theorem 5.2. Then for a second category subset of Kähler forms $\check{\omega}$ in the Kähler class of ω, the moduli space $\check{\mathcal{M}}_X$ of compact SL m-folds $\hat{X}$ with conical singularities in $(M, J, \check{\omega}, \Omega)$ isotopic to X consists solely of transverse $\hat{X}$, and so is a manifold of dimension* $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$.

A partial proof of this is given in [19, §9]. If we could treat the moduli spaces $\mathcal{M}_X$ as compact, the conjecture would follow from [19, Th. 9.3]. However, without knowing $\mathcal{M}_X$ is compact, the condition that $\check{\mathcal{M}}_X$ is smooth everywhere is in effect the intersection of an infinite number of genericity conditions on $\check{\omega}$, and we do not know that this intersection is dense (or even nonempty) in the Kähler class.

Notice that Conjecture 5.6 constrains the topology and cones of SL m-folds X with conical singularities that can occur in a generic almost Calabi–Yau m-fold, as we must have $\dim \mathcal{I}_{X'} \geqslant \dim \mathcal{O}_{X'}$.

6 Asymptotically Conical SL m-folds

We now discuss *Asymptotically Conical* SL m-folds L in $\mathbb{C}^m$, [18, Def. 7.1].

Definition 6.1. Let C be a closed SL cone in $\mathbb{C}^m$ with isolated singularity at 0 for $m > 2$, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$-manifold, not necessarily connected. Let g_Σ be the metric on Σ induced by the metric g' on $\mathbb{C}^m$ in (1), and r the radius function on $\mathbb{C}^m$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g') = r^2 g_\Sigma + \mathrm{d}r^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m-fold in $\mathbb{C}^m$. We call L *Asymptotically Conical (AC)* with *rate* $\lambda < 2$ and *cone* C if there exists a compact subset $K \subset L$

and a diffeomorphism $\varphi : \Sigma \times (T, \infty) \to L \setminus K$ for $T > 0$, such that

$$\left|\nabla^k(\varphi - \iota)\right| = O(r^{\lambda-1-k}) \quad \text{as } r \to \infty \text{ for } k = 0, 1.$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$.

This is very similar to Definition 3.7, and in fact there are strong parallels between the theories of SL m-folds with conical singularities and of Asymptotically Conical SL m-folds. We continue to assume $m > 2$ throughout.

6.1 Regularity and deformation theory of AC SL m-folds

Here are the analogues of Theorems 4.1 and 4.2, proved in [18, Th.s 7.4 & 7.11].

Theorem 6.2. *Suppose L is an AC SL m-fold in $\mathbb{C}^m$ with cone C, and let Σ, ι be as in Definition 6.1. Then for sufficiently large $T > 0$ there exist unique K, φ satisfying the conditions of Definition 6.1 and $(\varphi - \iota)(\sigma, r) \perp T_{\iota(\sigma,r)}C$ in $\mathbb{C}^m$ for all $(\sigma, r) \in \Sigma \times (T, \infty)$.*

Theorem 6.3. *In Theorem 6.2, if either $\lambda = \lambda'$, or λ, λ' lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$, then L is an AC SL m-fold with rate λ' and $\left|\nabla^k(\varphi - \iota)\right| = O(r^{\lambda'-1-k})$ for all $k \geqslant 0$. Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$ on $\Sigma \times (T, \infty)$.*

The *deformation theory* of Asymptotically Conical SL m-folds in $\mathbb{C}^m$ has been studied independently by Pacini [30] and Marshall [25]. Pacini's results are earlier, but Marshall's are more complete.

Definition 6.4. Suppose L is an Asymptotically Conical SL m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda < 2$, as in Definition 6.1. Define the *moduli space $\mathcal{M}_L^\lambda$ of deformations of L with rate* λ to be the set of AC SL m-folds $\hat{L}$ in $\mathbb{C}^m$ with cone C and rate λ, such that $\hat{L}$ is diffeomorphic to L and isotopic to L as an Asymptotically Conical submanifold of $\mathbb{C}^m$. One can define a natural *topology* on $\mathcal{M}_L^\lambda$, in a similar way to the conical singularities case of [19, Def. 5.6].

Note that if L is an AC SL m-fold with rate λ, then it is *also* an AC SL m-fold with rate λ' for any $\lambda' \in [\lambda, 2)$. Thus we have defined a 1-*parameter family* of moduli spaces $\mathcal{M}_L^{\lambda'}$ for L, and not just one. Since we did not impose any condition on λ in Definition 6.1 analogous to (11) in the conical singularities case, it turns out that $\mathcal{M}_L^\lambda$ depends nontrivially on λ.

The following result can be deduced from Marshall [25, Th. 6.2.15] and [25, Table 5.1]. (See also Pacini [30, Th. 2 & Th. 3].) It implies conjectures by the author in [6, Conj. 2.12] and [13, §10.2].

Theorem 6.5. *Let L be an Asymptotically Conical SL m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda < 2$, and let $\mathcal{M}_L^\lambda$ be as in Definition 6.4. Set $\Sigma = C \cap \mathcal{S}^{2m-1}$, and let $\mathcal{D}_\Sigma, N_\Sigma$ be as in §3.1 and $b^k(L), b^k_{\mathrm{cs}}(L)$ as in §3.4. Then*

(a) *If* $\lambda \in (0,2) \setminus \mathcal{D}_\Sigma$ *then* $\mathcal{M}_L^\lambda$ *is a manifold with*

$$\dim \mathcal{M}_L^\lambda = b^1(L) - b^0(L) + N_\Sigma(\lambda).$$

Note that if $0 < \lambda < \min\bigl(\mathcal{D}_\Sigma \cap (0,\infty)\bigr)$ *then* $N_\Sigma(\lambda) = b^0(\Sigma)$.

(b) *If* $\lambda \in (2-m,0)$ *then* $\mathcal{M}_L^\lambda$ *is a manifold of dimension* $b^1_{\mathrm{cs}}(L) = b^{m-1}(L)$.

This is the analogue of Theorems 2.10 and 5.2 for AC SL m-folds. If $\lambda \in (2-m,2) \setminus \mathcal{D}_\Sigma$ then the deformation theory for L with rate λ is *unobstructed* and $\mathcal{M}_L^\lambda$ is a *smooth manifold* with a given dimension. This is similar to the case of nonsingular compact SL m-folds in Theorem 2.10, but different to the conical singularities case in Theorem 5.2.

6.2 Cohomological invariants of AC SL m-folds

Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Using the notation of §3.4, as in (13) there is a long exact sequence

$$\cdots \to H^k_{\mathrm{cs}}(L,\mathbb{R}) \to H^k(L,\mathbb{R}) \to H^k(\Sigma,\mathbb{R}) \to H^{k+1}_{\mathrm{cs}}(L,\mathbb{R}) \to \cdots. \tag{19}$$

Following [18, Def. 7.2] we define *cohomological invariants* $Y(L), Z(L)$ of L.

Definition 6.6. Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\omega', \operatorname{Im}\Omega'$ in (1) are closed forms with $\omega'|_L \equiv \operatorname{Im}\Omega'|_L \equiv 0$, they define classes in the relative de Rham cohomology groups $H^k(\mathbb{C}^m; L,\mathbb{R})$ for $k = 2, m$. But for $k > 1$ we have the exact sequence

$$0 = H^{k-1}(\mathbb{C}^m,\mathbb{R}) \to H^{k-1}(L,\mathbb{R}) \xrightarrow{\cong} H^k(\mathbb{C}^m; L,\mathbb{R}) \to H^k(\mathbb{C}^m,\mathbb{R}) = 0.$$

Let $Y(L) \in H^1(\Sigma,\mathbb{R})$ be the image of $[\omega']$ in $H^2(\mathbb{C}^m; L,\mathbb{R}) \cong H^1(L,\mathbb{R})$ under $H^1(L,\mathbb{R}) \to H^1(\Sigma,R)$ in (19), and $Z(L) \in H^{m-1}(\Sigma,\mathbb{R})$ be the image of $[\operatorname{Im}\Omega']$ in $H^m(\mathbb{C}^m; L,\mathbb{R}) \cong H^{m-1}(L,\mathbb{R})$ under $H^{m-1}(L,\mathbb{R}) \to H^{m-1}(\Sigma,R)$ in (19).

Here are some conditions for $Y(L)$ or $Z(L)$ to be zero, [18, Prop. 7.3].

Proposition 6.7. *Let* L *be an AC SL* m*-fold in* $\mathbb{C}^m$ *with cone* C *and rate* λ*, and let* $\Sigma = C \cap \mathcal{S}^{2m-1}$*. If* $\lambda < 0$ *or* $b^1(L) = 0$ *then* $Y(L) = 0$*. If* $\lambda < 2-m$ *or* $b^0(\Sigma) = 1$ *then* $Z(L) = 0$.

6.3 Examples

Examples of AC SL m-folds L are constructed by Harvey and Lawson [3, §III.3], the author [7,8,9,11], and others. Nearly all the known examples (up to translations) have minimum rate λ either 0 or $2-m$, which are topologically significant values by Proposition 6.7. For instance, all examples in [8] have $\lambda = 0$, and [7, Th. 6.4] constructs AC SL m-folds with $\lambda = 2-m$ in $\mathbb{C}^m$ from any SL cone C in $\mathbb{C}^m$. The only explicit, nontrivial examples known to the author with $\lambda \neq 0, 2-m$ are in [9, Th. 11.6], and have $\lambda = \frac{3}{2}$.

We shall give three families of examples of AC SL m-folds L in $\mathbb{C}^m$ explicitly. The first family is adapted from Harvey and Lawson [3, §III.3.A].

Example 6.8. Let C_{HL}^m be the SL cone in $\mathbb{C}^m$ of Example 3.5. We shall define a family of AC SL m-folds in $\mathbb{C}^m$ with cone C_{HL}^m. Let $a_1, \ldots, a_m \geqslant 0$ with exactly two of the a_j zero and the rest positive. Write $\mathbf{a} = (a_1, \ldots, a_m)$, and define

$$\begin{aligned} L_{\text{HL}}^{\mathbf{a}} = \bigl\{ &(z_1, \ldots, z_m) \in \mathbb{C}^m : i^{m+1} z_1 \cdots z_m \in [0, \infty), \\ &|z_1|^2 - a_1 = \cdots = |z_m|^2 - a_m \bigr\}. \end{aligned} \tag{20}$$

Then $L_{\text{HL}}^{\mathbf{a}}$ is an AC SL m-fold in $\mathbb{C}^m$ diffeomorphic to $T^{m-2} \times \mathbb{R}^2$, with cone C_{HL}^m and rate 0. It is invariant under the $\mathrm{U}(1)^{m-1}$ group (8). It is surprising that equations of the form (20) should define a nonsingular submanifold of $\mathbb{C}^m$ *without boundary*, but in fact they do.

Now suppose for simplicity that $a_1, \ldots, a_{m-2} > 0$ and $a_{m-1} = a_m = 0$. As $\Sigma_{\text{HL}}^m \cong T^{m-1}$ we have $H^1(\Sigma_{\text{HL}}^m, \mathbb{R}) \cong \mathbb{R}^{m-1}$, and calculation shows that $Y(L_{\text{HL}}^{\mathbf{a}}) = (\pi a_1, \ldots, \pi a_{m-2}, 0) \in \mathbb{R}^{m-1}$ in the natural coordinates. Since $L_{\text{HL}}^{\mathbf{a}} \cong T^{m-2} \times \mathbb{R}^2$ we have $H^1(L_{\text{HL}}^{\mathbf{a}}, \mathbb{R}) = \mathbb{R}^{m-2}$, and $Y(L_{\text{HL}}^{\mathbf{a}})$ lies in the image $\mathbb{R}^{m-2} \subset \mathbb{R}^{m-1}$ of $H^1(L_{\text{HL}}^{\mathbf{a}}, \mathbb{R})$ in $H^1(\Sigma_{\text{HL}}^m, \mathbb{R})$, as in Definition 6.6. As $b^0(\Sigma_{\text{HL}}^m) = 1$, Proposition 6.7 shows that $Z(L_{\text{HL}}^{\mathbf{a}}) = 0$.

Take $C = C_{\text{HL}}^m$, $\Sigma = \Sigma_{\text{HL}}^m$ and $L = L_{\text{HL}}^{\mathbf{a}}$ in Theorem 6.5, and let $0 < \lambda < \min\bigl(\mathcal{D}_\Sigma \cap (0, \infty)\bigr)$. Then $b^1(L) = m - 2$, $b^0(L) = 1$ and $N_\Sigma(\lambda) = b^0(\Sigma) = 1$, so part (a) of Theorem 6.5 shows that $\dim \mathcal{M}_L^\lambda = m - 2$. This is consistent with the fact that L depends on $m - 2$ real parameters $a_1, \ldots, a_{m-2} > 0$.

The family of all $L_{\text{HL}}^{\mathbf{a}}$ has $\frac{1}{2} m(m-1)$ connected components, indexed by which two of $a_1, \ldots, a_m$ are zero. Using the theory of §7, these can give many *topologically distinct* ways to desingularize SL m-folds with conical singularities with these cones.

Our second family, from [7, Ex. 9.4], was chosen as it is easy to write down.

Example 6.9. Let $m, a_1, \ldots, a_m, k$ and $L_0^{a_1, \ldots, a_m}$ be as in Example 3.6. For $0 \neq c \in \mathbb{R}$ define

$$\begin{aligned} L_c^{a_1, \ldots, a_m} = \bigl\{ &\bigl(i e^{i a_1 \theta} x_1, e^{i a_2 \theta} x_2, \ldots, e^{i a_m \theta} x_m \bigr) : \theta \in [0, 2\pi), \\ &x_1, \ldots, x_m \in \mathbb{R}, \qquad a_1 x_1^2 + \cdots + a_m x_m^2 = c \bigr\}. \end{aligned}$$

Then $L_c^{a_1, \ldots, a_m}$ is an AC SL m-fold in $\mathbb{C}^m$ with rate 0 and cone $L_0^{a_1, \ldots, a_m}$. It is diffeomorphic as an immersed SL m-fold to $(\mathcal{S}^{k-1} \times \mathbb{R}^{m-k} \times \mathcal{S}^1)/\mathbb{Z}_2$ if $c > 0$, and to $(\mathbb{R}^k \times \mathcal{S}^{m-k-1} \times \mathcal{S}^1)/\mathbb{Z}_2$ if $c < 0$.

Our third family was first found by Lawlor [23], made more explicit by Harvey [2, p. 139–140], and discussed from a different point of view by the author in [8, §5.4(b)]. Our treatment is based on that of Harvey.

Example 6.10. Let $m > 2$ and $a_1, \ldots, a_m > 0$, and define polynomials p, P by

$$p(x) = (1 + a_1 x^2) \cdots (1 + a_m x^2) - 1 \quad \text{and} \quad P(x) = \frac{p(x)}{x^2}.$$

Define real numbers $\phi_1, \ldots, \phi_m$ and A by

$$\phi_k = a_k \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1 + a_k x^2)\sqrt{P(x)}} \quad \text{and} \quad A = \omega_m (a_1 \cdots a_m)^{-1/2}, \tag{21}$$

where ω_m is the volume of the unit sphere in $\mathbb{R}^m$. Clearly $\phi_k, A > 0$. But writing $\phi_1 + \cdots + \phi_m$ as one integral gives

$$\phi_1 + \cdots + \phi_m = \int_0^\infty \frac{p'(x)\mathrm{d}x}{(p(x)+1)\sqrt{p(x)}} = 2\int_0^\infty \frac{\mathrm{d}w}{w^2+1} = \pi,$$

making the substitution $w = \sqrt{p(x)}$. So $\phi_k \in (0,\pi)$ and $\phi_1 + \cdots + \phi_m = \pi$. This yields a 1-1 correspondence between m-tuples $(a_1, \ldots, a_m)$ with $a_k > 0$, and $(m+1)$-tuples $(\phi_1, \ldots, \phi_m, A)$ with $\phi_k \in (0,\pi)$, $\phi_1+\cdots+\phi_m = \pi$ and $A > 0$.

For $k = 1, \ldots, m$ and $y \in \mathbb{R}$, define a function $z_k : \mathbb{R} \to \mathbb{C}$ by

$$z_k(y) = \mathrm{e}^{i\psi_k(y)}\sqrt{a_k^{-1} + y^2}, \quad \text{where} \quad \psi_k(y) = a_k \int_{-\infty}^y \frac{\mathrm{d}x}{(1+a_k x^2)\sqrt{P(x)}}.$$

Now write $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$, and define a submanifold $L^{\boldsymbol{\phi},A}$ in $\mathbb{C}^m$ by

$$L^{\boldsymbol{\phi},A} = \big\{(z_1(y)x_1, \ldots, z_m(y)x_m) : y \in \mathbb{R},\ x_k \in \mathbb{R},\ x_1^2 + \cdots + x_m^2 = 1\big\}.$$

Then $L^{\boldsymbol{\phi},A}$ is closed, embedded, and diffeomorphic to $\mathcal{S}^{m-1}\times\mathbb{R}$, and Harvey [2, Th. 7.78] shows that $L^{\boldsymbol{\phi},A}$ is *special Lagrangian*. One can also show that $L^{\boldsymbol{\phi},A}$ is *Asymptotically Conical*, with rate $2-m$ and cone the union $\Pi^0 \cup \Pi^{\boldsymbol{\phi}}$ of two special Lagrangian m-planes $\Pi^0, \Pi^{\boldsymbol{\phi}}$ in $\mathbb{C}^m$ given by

$$\Pi^0 = \big\{(x_1, \ldots, x_m) : x_j \in \mathbb{R}\big\} \text{ and } \Pi^{\boldsymbol{\phi}} = \big\{(\mathrm{e}^{i\phi_1}x_1, \ldots, \mathrm{e}^{i\phi_m}x_m) : x_j \in \mathbb{R}\big\}.$$

As $\lambda = 2 - m < 0$ we have $Y(L^{\boldsymbol{\phi},A}) = 0$ by Proposition 6.7. Now $L^{\boldsymbol{\phi},A} \cong \mathcal{S}^{m-1}\times\mathbb{R}$ so that $H^{m-1}(L^{\boldsymbol{\phi},A},\mathbb{R}) \cong \mathbb{R}$, and $\Sigma = (\Pi^0 \cup \Pi^{\boldsymbol{\phi}}) \cap \mathcal{S}^{2m-1}$ is the disjoint union of two unit $(m-1)$-spheres $\mathcal{S}^{m-1}$, so $H^{m-1}(\Sigma,\mathbb{R}) \cong \mathbb{R}^2$. The image of $H^{m-1}(L^{\boldsymbol{\phi},A},\mathbb{R})$ in $H^{m-1}(\Sigma,\mathbb{R})$ is $\big\{(x,-x) : x \in \mathbb{R}\big\}$ in the natural coordinates. Calculation shows that $Z(L^{\boldsymbol{\phi},A}) = (A,-A) \in H^{m-1}(\Sigma,\mathbb{R})$, which is why we defined A this way in (21).

Apply Theorem 6.5 with $L = L^{\boldsymbol{\phi},A}$ and $\lambda \in (2-m, 0)$. As $L \cong \mathcal{S}^{m-1}\times\mathbb{R}$ we have $b^1_{\mathrm{cs}}(L) = 1$, so part (b) of Theorem 6.5 shows that $\dim \mathcal{M}^\lambda_{\scriptscriptstyle L} = 1$. This is consistent with the fact that when $\boldsymbol{\phi}$ is fixed, $L^{\boldsymbol{\phi},A}$ depends on one real parameter $A > 0$. Here $\boldsymbol{\phi}$ is fixed in $\mathcal{M}^\lambda_{\scriptscriptstyle L}$ as the cone $C = \Pi^0 \cup \Pi^{\boldsymbol{\phi}}$ of L depends on $\boldsymbol{\phi}$, and all $\hat{L} \in \mathcal{M}^\lambda_{\scriptscriptstyle L}$ have the same cone C, by definition.

7 Desingularizing singular SL m-folds

We now discuss the work of [20,21] on *desingularizing* compact SL m-folds with conical singularities. Here is the basic idea. Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, and X a compact SL m-fold in M with conical singularities $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Suppose $L_1, \ldots, L_n$ are AC SL m-folds in $\mathbb{C}^m$ with the same cones $C_1, \ldots, C_n$ as X.

If $t > 0$ then $tL_i = \{t\,\mathbf{x} : \mathbf{x} \in L_i\}$ is also an AC SL m-fold with cone C_i. We construct a 1-parameter family of compact, nonsingular *Lagrangian* m-folds N^t in (M, ω) for $t \in (0, \delta)$ by gluing tL_i into X at x_i, using a partition of unity.

When t is small, N^t is *close to special Lagrangian* (its phase is nearly constant), but also *close to singular* (it has large curvature and small injectivity radius). We prove using analysis that for small $t \in (0,\delta)$ we can deform N^t to a *special* Lagrangian m-fold $\tilde N^t$ in M, using a small Hamiltonian deformation.

The proof involves a delicate balancing act, showing that the advantage of being close to special Lagrangian outweighs the disadvantage ofbeing nearly singular. Doing this in full generality is rather complex. Here is our simplest desingularization result, [20, Th. 6.13].

Theorem 7.1. *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose $\lambda_i<0$ for $i=1,\ldots,n$, and $X'=X\setminus\{x_1,\ldots,x_n\}$ is connected.*

Then there exists $\epsilon>0$ and a smooth family $\bigl\{\tilde N^t : t\in(0,\epsilon]\bigr\}$ of compact, nonsingular SL m-folds in (M,J,ω,Ω), such that $\tilde N^t$ is constructed by gluing tL_i into X at x_i for $i=1,\ldots,n$. In the sense of currents, $\tilde N^t\rightarrow X$ as $t\rightarrow 0$.

The theorem contains two *simplifying assumptions*:

(a) that X' is connected, and

(b) that $\lambda_i<0$ for all i.

These avoid two kinds of *obstructions* to desingularizing X using the L_i.

In [20, Th. 7.10] we remove assumption (a), allowing X' not connected.

Theorem 7.2. *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Define $\psi: M\rightarrow(0,\infty)$ as in (3). Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose $\lambda_i<0$ for $i=1,\ldots,n$. Write $X'=X\setminus\{x_1,\ldots,x_n\}$ and $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$.*

Set $q=b^0(X')$, and let $X'_1,\ldots,X'_q$ be the connected components of X'. For $i=1,\ldots,n$ let $l_i=b^0(\Sigma_i)$, and let $\Sigma_i^1,\ldots,\Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j)=1,\ldots,q$ by $\Upsilon_i\circ\varphi_i\bigl(\Sigma_i^j\times(0,R')\bigr)\subset X'_{k(i,j)}$ for $i=1,\ldots,n$ and $j=1,\ldots,l_i$. Suppose that

$$\sum_{\substack{1\leqslant i\leqslant n,\;1\leqslant j\leqslant l_i:\\ k(i,j)=k}}\psi(x_i)^m Z(L_i)\cdot[\Sigma_i^j]=0 \quad \text{for all } k=1,\ldots,q. \tag{22}$$

Suppose also that the compact m-manifold N obtained by gluing L_i into X' at x_i for $i=1,\ldots,n$ is connected. A sufficient condition for this to hold is that X and L_i for $i=1,\ldots,n$ are connected.

Then there exists $\epsilon>0$ and a smooth family $\bigl\{\tilde N^t : t\in(0,\epsilon]\bigr\}$ of compact, nonsingular SL m-folds in (M,J,ω,Ω) diffeomorphic to N, such that $\tilde N^t$ is constructed by gluing tL_i into X at x_i for $i=1,\ldots,n$. In the sense of currents in Geometric Measure Theory, $\tilde N^t\rightarrow X$ as $t\rightarrow 0$.

The new issue here is that if X' is not connected then there is an *analytic obstruction* to deforming N^t to $\tilde N^t$, because the Laplacian Δ^t on functions on N^t has *small eigenvalues* of size $O(t^{m-2})$. As in §6.2 the L_i have *cohomological invariants* $Z(L_i)$ in $H^{m-1}(\Sigma_i,\mathbb{R})$ derived from the relative cohomology class of $\operatorname{Im}\Omega'$. It turns out that we can only deform N^t to $\tilde N^t$ if the $Z(L_i)$ satisfy (22). This equation arises by requiring the projection of an error term to the eigenspaces of Δ^t with small eigenvalues to be zero.

In [21, Th. 6.13] we remove assumption (b), extending Theorem 7.1 to the case $\lambda_i \leqslant 0$, and allowing $Y(L_i)\neq 0$.

Theorem 7.3. *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold for $2<m<6$, and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$. Let $L_1,\ldots,L_n$ be Asymptotically Conical SL m-folds in $\mathbb{C}^m$ with cones $C_1,\ldots,C_n$ and rates $\lambda_1,\ldots,\lambda_n$. Suppose that $\lambda_i\leqslant 0$ for $i=1,\ldots,n$, that $X'=X\setminus\{x_1,\ldots,x_n\}$ is connected, and that there exists $\varrho\in H^1(X',\mathbb{R})$ such that $\bigl(Y(L_1),\ldots,Y(L_n)\bigr)$ is the image of ϱ under the map $H^1(X',\mathbb{R})\rightarrow\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ in (13), where $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$.*

Then there exists $\epsilon>0$ and a smooth family $\bigl\{\tilde N^t : t\in(0,\epsilon]\bigr\}$ of compact, nonsingular SL m-folds in (M,J,ω,Ω), such that $\tilde N^t$ is constructed by gluing tL_i into X at x_i for $i=1,\ldots,n$. In the sense of currents, $\tilde N^t\rightarrow X$ as $t\rightarrow 0$.

From §6.3, the L_i have *cohomological invariants* $Y(L_i)$ in $H^1(\Sigma_i,\mathbb{R})$ derived from the relative cohomology class of ω'. The new issue in Theorem 7.3 is that if $Y(L_i)\neq 0$ then there are obstructions to the existence of N^t as a *Lagrangian* m-fold. That is, we can only define N^t if the $Y(L_i)$ satisfy an equation. This did not appear in Theorem 7.1, as $\lambda_i<0$ implies that $Y(L_i)=0$.

To define the N^t when $Y(L_i)\neq 0$ we must also use a more complicated construction. This introduces new errors. To overcome these errors when we deform N^t to $\tilde N^t$ we must assume that $m<6$. There is also [21, Th. 6.12] a result combining the modifications of Theorems 7.2 and 7.3, but for brevity we will not give it.

8 Directions for future research

Finally we discuss directions the field of special Lagrangian singularities might develop in the future, giving a number of problems the author believes are worth attention. Some of these problems may be too difficult to solve completely, but can still serve as a guide.

8.1 The index of singularities of SL m-folds

We now consider the *boundary* $\partial\mathcal{M}_N$ of a moduli space $\mathcal{M}_N$ of SL m-folds.

Definition 8.1. Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, N a compact, nonsingular SL m-fold in M, and $\mathcal{M}_N$ the moduli space of deformations of N in M. Then $\mathcal{M}_N$ is a smooth manifold of dimension $b^1(N)$, in general noncompact. We can construct a natural *compactification* $\overline{\mathcal{M}}_N$ as follows.

Regard $\mathcal{M}_N$ as a moduli space of special Lagrangian *integral currents* in the sense of Geometric Measure Theory, as discussed in [18, §6]. Let $\overline{\mathcal{M}}_N$ be the closure of $\mathcal{M}_N$ in the space of integral currents. As elements of $\mathcal{M}_N$ have uniformly bounded volume, $\overline{\mathcal{M}}_N$ is *compact*. Define the *boundary* $\partial\mathcal{M}_N$ to be $\overline{\mathcal{M}}_N \setminus \mathcal{M}_N$. Then elements of $\partial\mathcal{M}_N$ are *singular SL integral currents.*

In good cases, say if (M, J, ω, Ω) is suitably generic, it seems reasonable that $\partial\mathcal{M}_N$ should be divided into a number of *strata*, each of which is a moduli space of singular SL m-folds with singularities of a particular type, and is itself a manifold with singularities. In particular, some or all of these strata could be moduli spaces $\mathcal{M}_X$ of SL m-folds with isolated conical singularities, as in §5.

Suppose $\mathcal{M}_N$ is a moduli space of compact, nonsingular SL m-folds N in (M, J, ω, Ω), and $\mathcal{M}_X$ a moduli space of singular SL m-folds in $\partial\mathcal{M}_N$ with singularities of a particular type, and $X \in \mathcal{M}_X$. Following [22, §8.3], we (loosely) define the *index* of the singularities of X to be $\operatorname{ind}(X) = \dim\mathcal{M}_N - \dim\mathcal{M}_X$, provided $\mathcal{M}_X$ is smooth near X. Note that $\operatorname{ind}(X)$ depends on N as well as X.

In [22, Th. 8.10] we use the results of [19,20,21] to compute $\operatorname{ind}(X)$ when X is *transverse* with conical singularities, in the sense of Definition 5.4. Here is a simplified version of the result, where we assume that $H^1_{\rm cs}(L_i, \mathbb{R}) \to H^1(L_i, \mathbb{R})$ is surjective to avoid a complicated correction term to $\operatorname{ind}(X)$ related to the obstructions to defining N^t as a Lagrangian m-fold.

Theorem 8.2. *Let X be a compact, transverse SL m-fold in (M, J, ω, Ω) with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be AC SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$, such that the natural projection $H^1_{\rm cs}(L_i, \mathbb{R}) \to H^1(L_i, \mathbb{R})$ is surjective. Construct desingularizations N of X by gluing AC SL m-folds $L_1, \ldots, L_n$ in at $x_1, \ldots, x_n$, as in §7. Then*

$$\operatorname{ind}(X) = 1 - b^0(X') + \textstyle\sum_{i=1}^n b^1_{\rm cs}(L_i) + \sum_{i=1}^n \text{s-ind}(C_i). \tag{23}$$

If the cones C_i are not *rigid*, for instance if $C_i \setminus \{0\}$ is not connected, then (23) should be corrected, as in [22, §8.3]. If Conjecture 5.6 is true then for a *generic* Kähler form ω, *all* compact SL m-folds X with conical singularities are transverse, and so Theorem 8.2 and [22, Th. 8.10] allow us to calculate $\operatorname{ind}(X)$.

Now singularities with *small index* are the most commonly occurring, and so arguably the most interesting kinds of singularity. Also, as $\operatorname{ind}(X) \leqslant \dim\mathcal{M}_N$, for various problems, such as those in §8.3 and §8.4, it will only be necessary to know about singularities with index up to a certain value. This motivates the following:

Problem 8.3. Classify types of singularities of SL 3-folds with *small index* in suitably generic almost Calabi–Yau 3-folds, say with index 1,2 or 3.

Here we restrict to $m = 3$ to make the problem more feasible, though still difficult. Note, however, that we do *not* restrict to isolated conical singularities, so a complete, rigorous answer would require a theory of more general kinds of singularities of SL 3-folds.

One can make some progress on this problem simply by studying the many examples of singular SL 3-folds in [3,4] and [8,9,10,11,12,13,14,15,16,17], calculating or guessing the index of each, and ruling out other kinds of singularities by plausible-sounding arguments. Using these techniques I have a conjectural classification of index 1 singularities of SL 3-folds, which involves the SL T^2-cone $C^3_{\rm HL}$ of (7), and several different kinds of singularity whose tangent cone is two copies of $\mathbb{R}^3$, intersecting in 0, $\mathbb{R}$ or $\mathbb{R}^3$.

Coming from another direction, *integrable systems* techniques may yield rigorous classification results for SL T^2-cones by index. Haskins [5, Th. A] has used them to prove that the SL T^2-cone $C^3_{\rm HL}$ in $\mathbb{C}^3$ of (7) is up to SU(3) equivalence the *unique* SL T^2-cone C with s-ind$(C) = 0$. Now the index of a singularity modelled on C is at least s-ind$(C) + 1$, so this implies that $C^3_{\rm HL}$ is the unique SL T^2-cone with index 1 in Problem 8.3.

8.2 Singularities which are not isolated conical

Singularities of SL m-folds which are not 'isolated conical singularities' in the sense of Definition 3.7 are an important, but virtually unexplored, subject. Here are some known classes of nontrivial examples when $m = 3$.

(i) In [11] we study *ruled* SL 3-folds in $\mathbb{C}^3$, that is, SL 3-folds N fibred by a 2-dimensional family Σ of real straight lines in $\mathbb{C}^3$. When Σ is nonsingular N can still have singularities, and examples may be written down very explicitly, as in [11, Th. 7.1].

The *tangent cones* of such singularities, in the sense of Geometric Measure Theory, are generally $\mathbb{R}^3$ with multiplicity $k > 1$. Near the singular point, the SL 3-fold resembles a k-fold branched cover of $\mathbb{R}^3$, branched along $\mathbb{R}$. A similar class of singularities of SL 3-folds, with tangent cone $\mathbb{R}^3$ with multiplicity 2, is studied in [9, §6].

(ii) In [14,15,16] we study SL 3-folds in $\mathbb{C}^3$ invariant under the U(1)-action

$$\mathrm{e}^{i\theta} : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{-i\theta} z_2, z_3) \quad \text{for } \mathrm{e}^{i\theta} \in \mathrm{U}(1).$$

The three papers are surveyed in [17]. A U(1)-invariant SL 3-fold N may locally be written in the form

$$N = \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y), \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x,y) \in S\big\},$$

where S is a domain in $\mathbb{R}^2$, $a \in \mathbb{R}$ and $u, v : S \to \mathbb{R}$ satisfy (in a weak sense if $a = 0$) the *nonlinear Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2 + a^2\big)^{1/2} \frac{\partial u}{\partial y}. \tag{24}$$

Using analytic techniques, we construct and study solutions u, v of (24) satisfying boundary conditions on a strictly convex domain S. These include many *singular solutions*, and we show in [16, §9–§10] that we can construct countably many distinct geometrical-topological types of isolated SL 3-fold singularities, whose tangent cone is the union of two $\mathbb{R}^3$'s in $\mathbb{C}^3$, intersecting in $\mathbb{R}$.

There appear to the author to be two ways of studying special Lagrangian singularities which are not isolated conical. The first is to try and study *all* singularities of special Lagrangian integral currents, using Geometric Measure Theory. As far as the author understands (which is not very far), it will be difficult to use the special Lagrangian condition in GMT, or to say anything nontrivial about special Lagrangian singularities in this generality.

The second way is to define some restricted class of singularities and then study them, just as we did in §3–§7. The problem here is to decide upon a suitable kind of *local model* for the singularities, and appropriate *asymptotic conditions* for how the SL m-fold approaches the local model near the singularity. Now not just any local model and asymptotic conditions will do.

For a class of singularities to be worth studying, they should occur reasonably often in 'real life', so that, for instance, examples of such singularities might occur in compact SL m-folds in fairly generic almost Calabi–Yau m-folds. A good test of this is whether the *deformation theory* of compact SL m-folds with this kind of singularity is well-behaved. That is, the analogue of Theorem 5.2 should hold, with *finite-dimensional* obstruction space $\mathcal{O}_{X'}$.

One very obvious way to make examples of SL m-folds with nonisolated singularities is to consider $C \times \mathbb{R}^{m-k}$ in $\mathbb{C}^k \times \mathbb{C}^{m-k} = \mathbb{C}^m$, where C is an SL cone in $\mathbb{C}^k$ with isolated singularity at 0, and $3 \leqslant k < m$. So we could study SL m-folds with singularities locally modelled on $C \times \mathbb{R}^{m-k}$. Calculations by the author indicate that the deformation theory of such singular SL m-folds will be well-behaved if and only if C is *stable*. Therefore we propose:

Problem 8.4. Let $3 \leqslant k < m$, and suppose C is an SL cone in $\mathbb{C}^k$ with an isolated singularity at 0 which is *stable*, in the sense of Definition 3.4. Study compact SL m-folds N in almost Calabi–Yau m-folds (M, J, ω, Ω), where the singular set S of N is a compact $(m-k)$-submanifold of M, and N is modelled on $C \times \mathbb{R}^{m-k}$ in $\mathbb{C}^k \times \mathbb{C}^{m-k} = \mathbb{C}^m$ at each singular point $s \in S$.

Here we have not defined what we mean by 'modelled on'. There should be some fairly natural asymptotic condition, along the lines of (12). Perhaps, as in Theorem 4.3, it will be equivalent to N having tangent cone $C \times \mathbb{R}^{m-k}$ with multiplicity 1 at each $s \in S$.

A related problem is to classify the possible stable C:

Problem 8.5. Classify special Lagrangian cones C in $\mathbb{C}^m$ for $m \geqslant 3$ with an isolated singularity at 0 which are *stable*, in the sense of Definition 3.4.

As above, by Haskins [5, Th. A] the SL T^2-cone C^3_{HL} in $\mathbb{C}^3$ of (7) is up to SU(3) equivalence the *unique* stable SL T^2-cone C in $\mathbb{C}^3$. In fact C^3_{HL} is the *only* example of a stable SL cone in $\mathbb{C}^m$ for $m \geqslant 3$ known to the author. It is conceivable that it really is the only example, so that the answer to Problem 8.5 is C^3_{HL} and no others.

We can also look for other interesting classes of singularities with well-behaved deformation theory. The key is to find suitable asymptotic conditions.

Problem 8.6. Let C be an SL cone in $\mathbb{C}^m$ with nonisolated singularity at 0, or with multiplicity $k > 1$. Can you find a good, natural set of asymptotic conditions for SL m-folds with isolated singularities with tangent cone C?

One way to approach this is through *examples*: we find some class of examples of singular SL m-folds, calculate their asymptotic behaviour near their singularities, and try and abstract the important features. For the examples in (i) above this may be easy, as they are very explicit. But for those in (ii) above the author failed miserably to understand the asymptotic behaviour.

8.3 The SYZ Conjecture

Mirror Symmetry is a mysterious relationship between pairs of Calabi–Yau 3-folds $M, \hat{M}$, arising from a branch of physics known as *String Theory*, and leading to some very strange and exciting conjectures about Calabi–Yau 3-folds.

Roughly speaking, String Theorists believe that each Calabi–Yau 3-fold M has a quantization, a *Super Conformal Field Theory* (SCFT). If $M, \hat{M}$ have SCFT's isomorphic under a certain simple involution of SCFT structure, we say that $M, \hat{M}$ are *mirror* Calabi–Yau 3-folds. One can argue using String Theory that $H^{1,1}(M) \cong H^{2,1}(\hat{M})$ and $H^{2,1}(M) \cong H^{1,1}(\hat{M})$. The mirror transform also exchanges things related to the complex structure of M with things related to the symplectic structure of $\hat{M}$, and vice versa.

The *SYZ Conjecture*, due to Strominger, Yau and Zaslow [31] in 1996, gives a geometric explanation of Mirror Symmetry. Here is an attempt to state it.

Conjecture 8.7 (Strominger–Yau–Zaslow). *Suppose M and $\hat{M}$ are mirror Calabi–Yau 3-folds. Then (under some additional conditions) there should exist a compact topological 3-manifold B and surjective, continuous maps $f : M \to B$ and $\hat{f} : \hat{M} \to B$ with fibres $X_b = f^{-1}(b)$ and $\hat{X}_b = \hat{f}^{-1}(b)$ for $b \in B$, such that*

(i) *There exists a dense open set $B_0 \subset B$, such that for each $b \in B_0$, the fibres $X_b, \hat{X}_b$ are nonsingular special Lagrangian 3-tori T^3 in M and $\hat{M}$, which are in some sense dual to one another.*

(ii) *For each $b \in \Delta = B \setminus B_0$, the fibres X_b, $\hat{X}_b$ are expected to be singular special Lagrangian 3-folds in M and $\hat{M}$.*

We call $f, \hat{f}$ *special Lagrangian fibrations*, and the set of singular fibres Δ is called the *discriminant*. It is not yet clear what the final form of the SYZ Conjecture should be. Much work has been done on it, working primarily with *Lagrangian* fibrations, by authors such as Mark Gross and Wei-Dong Ruan. For references see [10].

The author's approach to the SYZ Conjecture, focussing primarily on *special Lagrangian singularities*, is set out in [10], and we do not have space to discuss it here. Very briefly, we argue that for generic (almost) Calabi–Yau 3-folds (ii) will not hold, as the discriminants $\Delta, \hat{\Delta}$ of $f, \hat{f}$ cannot be homeomorphic near certain kinds of singular fibre. We also suggest that the final form of the SYZ Conjecture should be an *asymptotic statement* about 1-*parameter families* of Calabi-Yau 3-folds approaching the *large complex structure limit.*

Problem 8.8. Study *special Lagrangian fibrations* $f : M \to B$ of almost Calabi–Yau 3-folds (M, J, ω, Ω), particularly when ω is *generic* in its Kähler class. Clarify/prove/disprove the SYZ Conjecture.

Note that the ideas of §8.1 will be helpful here. As B has dimension 3, we see that $\mathrm{ind}(X_b) \leqslant 3$ for all $b \in \Delta$. If Conjecture 5.6 holds, ω is generic, and $f^{-1}(b)$ has isolated conical singularities, then X_b is *transverse.* We can then use Theorem 8.2 or [22, Th. 8.10] to calculate $\mathrm{ind}(X_b)$, and $\mathrm{ind}(X_b) \leqslant 3$ will severely restrict the possible singular behaviour.

8.4 Invariants from counting SL homology spheres

In [6] the author proposed to define an invariant of almost Calabi–Yau 3-folds (M, J, ω, Ω) by counting special Lagrangian rational homology 3-spheres N (which occur in 0-dimensional moduli spaces) in a given homology class, with a certain topological weight. This invariant will only be interesting if it is conserved under deformations of the underlying almost Calabi–Yau 3-fold, or at least transforms in a rigid way as the cohomology classes $[\omega], [\Omega]$ change.

During such a deformation, nonsingular SL 3-folds can develop singularities and disappear, or new ones appear, which might change the invariant. In [6] the author showed that if we count rational SL homology spheres N with weight $\bigl|H_1(N, \mathbb{Z})\bigr|$, then under two kinds of singular behaviour of SL 3-folds, the resulting invariant is independent of $[\omega]$, and transforms according to certain rules as $[\Omega]$ crosses real hypersurfaces in complex structure moduli space where phases of $\alpha, \beta \in H_3(M, \mathbb{Z})$ become equal.

Again, the ideas of §8.1 will be helpful here. It is enough for us to study how the invariant changes along *generic* 1-*parameter families* of almost Calabi–Yau 3-folds. The only kinds of singularities of SL homology 3-spheres that arise in such families will have index 1. So if we can complete the index 1 classification in Problem 8.3, we should be able to resolve the conjectures of [6].

In fact, I now believe that interesting invariants of almost Calabi–Yau m-folds by 'counting' SL m-folds can be defined for all $m \geqslant 3$. The definition, properties and transformation laws of these invariants are formidably complex and difficult, even to state. The best approach I have to them is to use Homological Mirror Symmetry to translate the problem to the derived category $\mathcal{T} = D^b(\mathrm{Fuk}(M, \omega))$ of the Fukaya category of (M, ω).

Then SL m-folds conjecturally correspond to *stable objects* of the triangulated category $\mathcal{T}$, under a stability condition à la Tom Bridgeland. The invariants are Euler characteristics of moduli spaces of *configurations* in $\mathcal{T}$, which are finite collections of (stable or semistable) objects and morphisms in $\mathcal{T}$ satisfying some axioms. In this set-up, using algebra and category theory, I can rigorously develop the definition and properties of the invariants, and their transformation rules under change of stability condition (effectively, deformation of J, Ω). I am writing (yet) another series of papers about this.

Problem 8.9. Try to use moduli spaces of compact SL m-folds (possibly immersed, or singular) to define systems of invariants of an almost Calabi–Yau m-fold (M, J, ω, Ω) for $m \geqslant 3$. These invariants should be defined for ω generic in its Kähler class, and the key property we want is that they should be *independent of* ω. Compute the invariants for the quintic. Calculate the transformation rules for the invariants under deformation of J, Ω. Relate them to Homological Mirror Symmetry, and to 'branes' in String Theory.

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