Vector bundles for affine Lie algebras and the moduli space of curves

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1 Introduction

In these notes, I discuss a class of vector bundles on the moduli space of curves that are defined using representation theory. As I'll motivate and explain, my hope is that these bundles can help us answer some very basic open questions about $\overline{\mathrm{M}}_{g,n}$. I'll also tell you about a problem that we were able to answer about the fibers of the bundles using the moduli space of curves.

In Section 4, I discuss three related open problems.

2 The moduli space of curves

For the purpose of the talk, we consider the moduli space of curves as a projective variety, whose (closed) points correspond to (isomorphism classes) of stable n-pointed curves $x = (C; p_1, \ldots, p_n)$ of genus g. In particular, C has at worst simple nodal singularities, the marked points p_i are smooth points of C, and there are a finite number of automorphisms of x. It can be profitable to consider the stack $\overline{\mathcal{M}}_{g,n}$, and we briefly discuss this functorial point of view, and how one is led to pointed curves, even if mainly interested in smooth curves of genus g, in Section 5.

There are good reasons that the moduli space of curves is a widely studied object in algebraic geometry and related fields.

For instance

- it tells one about smooth curves and how they degenerate;

- it has emerged as somewhat of a prototype for what one may want to achieve when constructing a moduli space, guiding study of moduli spaces of higher dimensional varieties and of other objects such as sheaves of various sorts;
- because S_n acts on $\overline{M}_{g,n}$ by permuting the marked points, it has a combinatorial structure, which has the feel of a homogeneous variety.

This last feature, the combinatorial structure of $\overline{M}_{g,n}$, often allows one to reduce problems for moduli of higher genus curves to $\overline{M}_{0,n}$. But still very basic open questions remain. I'll illustrate this with the example about cones of nef divisors.

Definition 2.1. *A divisor D on a projective variety X is nef if $D \cdot C \geqslant 0$ for all curves C on X.*

Examples of nef divisors include $f^*\mathcal{O}_Y(1)$ for any morphism $f : X \to Y$, where Y is a projective variety.

Definition 2.2. $\mathrm{Nef}(X) = \{$ *nef divisors on* $X\}$.

The set $\mathrm{Nef}(X)$ is a cone, an invariant of X that tells us about morphisms from X to other projective varieties.

The first and most basic open question one can ask about $\mathrm{Nef}(X)$ is whether or not it is polyhedral. If yes, then the idea would be that one could identify all the maps admitted by the variety. If no, then maybe that would be too hard to do.

Nef cones of complete toric varieties are polyhedral. So one might expect this for varieties that are similar to toric varieties.

It is known that $\mathrm{Nef}(\overline{M}_{0,n})$ is polyhedral for $n \leqslant 7$ [KM13], and $\mathrm{Nef}(\overline{M}_g)$ is polyhedral for $g \leqslant 24$ [Gib09]. There is a conjecture that would imply polyhedrality of the nef cone $\overline{M}_{g,n}$. If the conjecture holds for $g = 0$ and all n then it holds for $\overline{M}_{g,n}$.

I give the details of this conjecture in my notes.

We set:

$$\delta^k(\overline{M}_{g,n}) = \{(C, \vec{p}) \in \overline{M}_{g,n} \ : \ C \text{ has at least k nodes } \}$$

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{M}_{g,n})$ are called F-Curves. One can ask the following question:

Conjecture 2.3. [GKM02] *A divisor on $\overline{M}_{g,n}$ is nef, if and only if it nonnegatively intersects all the* F*-Curves.*

In [GKM02], we showed that in fact a positive solution to this question for S_g-invariant nef divisors on $\overline{M}_{0,g+n}$ would give a positive answer for divisors on $\overline{M}_{g,n}$. In particular, the birational geometry of $\overline{\mathrm{M}}_{0,g}$ controls aspects of the birational geometry of $\overline{\mathrm{M}}_g$. We know now that the answer to this question is true on $\overline{\mathrm{M}}_{0,n}$ for $n \leqslant 7$ [KM13], and on $\overline{\mathrm{M}}_g$ for $g \leqslant 24$ [Gib09].

For the purpose of this talk, this example tells us that it is worthwhile studying nef divisors on $\overline{\mathrm{M}}_{0,n}$ even if we are interested in the general story. Vector bundles of covacua for affine Lie algebras give rise to elements of the cone of nef divisors: each bundle on $\overline{\mathrm{M}}_{0,n}$ is globally generated, and so has base point free first Chern class (ie. is of the form f^*A for some morphism $f : \overline{\mathrm{M}}_{0,n} \to Y$ where Y is a projective variety, and A is an ample line bundle on it). Fakhruddin proved that the set of first Chern classes for positive rank level one bundles for $\mathfrak{sl}_2$ gives a basis for the Picard group of $\overline{\mathrm{M}}_{0,n}$, and so even in the simplest case, we get a full dimensional sub-cone of the nef cone.

3 Vector bundles for affine Lie algebras

As I will outline below, given a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant integral weights for $\mathfrak{g}$ at level ℓ[1], one can construct a vector space associated to a stable n-pointed curve $(C; p_1, \dots, p_n)$ of genus g. These vector spaces fit together to form a a vector bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on the moduli space $\overline{\mathrm{M}}_{g,n}$.

For references for these bundles defined using affine Lie algebras, see [TUY89, ?Ueno, Bea96] and [Fak12]. A broader picture exists for conformal vertex operator algebras, as described in the book of Frenkel and Ben-Zvi [BzF01]. In case those CVOAs are regular chiral algebras, in [NT05], Nagatomo and Tsuchiya proved they satisfy "Factorization" and "Propagation of Vacua", two important properties satisfied by the vector bundles I'm talking about today.

3.1 Description of the fibers

3.1.1 Finite dimensional situation:

First we suppose that C is a smooth curve of genus g with at least one marked point. We'll take care of the other cases at the end.

Recall that to λ_i there corresponds a unique finite dimensional $\mathfrak{g}$-module V_{λ_i}. Set

[1]We say for $\lambda_i \in \mathcal{P}_\ell(\mathfrak{g})$.

$V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots V_{\lambda_n}$ and define an action

$$\mathfrak{g} \times V_{\vec{\lambda}} \to V_{\vec{\lambda}} \ \ (g, v_1 \otimes \cdots v_n) \mapsto \sum_{i=1}^{n} v_1 \otimes \cdots v_{i-1} \otimes (g \cdot v_i) \otimes v_{i+1} \otimes \cdots v_n.$$

We write $[V_{\vec{\lambda}}]_{\mathfrak{g}}$ for the **space of coinvariants of** $V_{\vec{\lambda}}$: The largest quotient of $V_{\vec{\lambda}}$ on which $\mathfrak{g}$ acts trivially. That is, the quotient of $V_{\vec{\lambda}}$ by the subspace spanned by the vectors $X \cdot v$ where $X \in \mathfrak{g}$ and $v \in V_{\vec{\lambda}}$.

Let V and W be two $\mathfrak{a}$-modules. The space of coinvariants $[V \otimes W]_{\mathfrak{g}}$ is equal to the quotient of $V \otimes W$ by the subspace spanned by the elements of the form

$$X v \otimes w + v \otimes X w,$$

where $X \in \mathfrak{g}$, $v \in V$, and $w \in W$.

3.1.2 Infinite dimensional analogues:

Given a smooth n-pointed curve $(C, \vec{p})$, to construct the fiber $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C, \vec{p}}$ we will use two new Lie algebras:

First, for each $i \in \{1, \dots\}$ we will use

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \ \mathbb{C} \cdot c,$$

where by $\mathbb{C}((\xi_i))$, we mean the field of Laurant power series over $\mathbb{C}$ in the variable ξ_i, and c is in the center of $\hat{\mathfrak{g}}_i$. To define the bracket, we note that elements in $\hat{\mathfrak{g}}_i$ are tuples $(a_i, \alpha c)$, with $a_i = \sum_j X_{ij} \otimes f_{ij}$, with $f_{ij} \in \mathbb{C}((\xi_i))$. We define the bracket on simple tensors:

$$[(X \otimes f, \alpha c), (Y \otimes g, \beta c)] = ([X, Y] \otimes fg, c(X, Y) \cdot \mathrm{Res}_{\xi_i = 0}(g(\xi_i) df(\xi_i))).$$

Checking $\hat{\mathfrak{g}}_i$ is a Lie algebra done in Section 6, where we also outline the construction of the infinite dimensional analogue H_{λ_i} of V_{λ_i}: It turns out that H_{λ_i} is a unique $\hat{\mathfrak{g}}_i$-module, although infinite dimensional.

Now for the second Lie algebra:

Let $U = C \setminus \{p_1, \dots, p_n\}$. Because C is smooth, and has at least one marked point, U is affine By $\mathfrak{g}(U)$ we mean the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_C(U)$.

Choose a local coordinate ξ_i at each point p_i, and denote by f_{p_i} the Laurant expansion of any element $f \in \mathcal{O}_C(U)$. Then for each i, we get a ring homomorphism

$$\mathcal{O}_C(U) \to \mathbb{C}((\xi_i)), \ \ f \mapsto f_{p_i},$$

and hence for each i, we obtain a map (this is not a Lie algebra embedding)

$$\mathfrak{g}(U) \to \hat{\mathfrak{g}}_i \quad \mathrm{X}\otimes f \mapsto (\mathrm{X}\otimes f_{p_i}, 0).$$

Set $\mathrm{H}_{\vec{\lambda}} = H_{\lambda_1} \otimes \cdots H_{\lambda_n}$ and define the following, which we will show is an action:

$$\mathfrak{g}(U) \times \mathrm{H}_{\vec{\lambda}} \to \mathrm{H}_{\vec{\lambda}} \;\; (g, w_1 \otimes \cdots w_n) \mapsto \sum_{i=1}^{n} w_1 \otimes \cdots w_{i-1} \otimes (g \cdot w_i) \otimes w_{i+1} \otimes \cdots w_n. \tag{1}$$

Claim 3.1. *Equation 1 defines an action of* $\mathfrak{g}(U)$ *on* $\mathrm{H}_{\vec{\lambda}}$.

Proof. Given $X \otimes f$, and $Y \otimes g \in \mathfrak{g}(U)$, and a simple tensor $v = v_1 \otimes \cdots \otimes v_n \in \mathrm{H}_{\vec{\lambda}}$, we want to check that

$$[X \otimes f, Y \otimes g] \cdot v = (X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big).$$

The right hand side simplifies as follows:

$$\begin{aligned}
&(X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big) \\
&\qquad = (X \otimes f) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&\qquad - (Y \otimes g) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&= \Big(\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (X \otimes f_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&- \Big(\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (Y \otimes g_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&\quad = \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot \big((Y \otimes g_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&\quad - \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot \big((X \otimes f_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\
&\qquad\quad = \Big(\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes \big([X, Y] + (fg)_{p_i}\big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)
\end{aligned} \tag{2}$$

The left hand side simplifies as follows:

$$\begin{aligned}
&\sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big([X, Y] \otimes f_{p_i} g_{p_i} + (X, Y) \operatorname{Res}_{\xi_i = 0} g_{p_i} d f_{p_i} c\Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \\
&\qquad = \sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big([X, Y] \otimes f_{p_i} g_{p_i}\Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \\
&\qquad\quad + \sum_{1 \leqslant i \leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big((X, Y) \operatorname{Res}_{\xi_i = 0} g_{p_i} d f_{p_i} c\Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n.
\end{aligned} \tag{3}$$

Now, by definition, $c \cdot v_i = \ell \cdot v_i$ for all i, and so we can rewrite the second summand as follows

$$\begin{aligned}(4)\quad &\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes ((X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} c) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \\ &= \sum_{1\leqslant i\leqslant n} (X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} \left(v_1 \otimes \cdots \otimes v_{i-1} \otimes c \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\right) \\ &= \sum_{1\leqslant i\leqslant n} (X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} \left(v_1 \otimes \cdots \otimes v_{i-1} \otimes \ell \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\right) \\ &= \Big(\ell \sum_{1\leqslant i\leqslant n} (X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i}\Big) \left(v_1 \otimes \cdots \otimes v_n\right).\end{aligned}$$

Since $\sum_{1\leqslant i\leqslant n}(X,Y)\operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} = 0$, this contribution is zero. Therefore the left and right hand sides of the expressions are the same, and we have checked that $\mathfrak{g}(U)$ acts on $\mathrm{H}_{\vec{\lambda}}$ as claimed. □

We now set

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} = [\mathrm{H}_{\vec{\lambda}}]_{\mathfrak{g}(U)}.$$

Now if C is a smooth curve with no marked points, one can use Propagation of Vacua (see Section 7) which says that $\mathbb{V}(\mathfrak{g}, \{0\}, \ell)$ is isomorphic to the pullback of $\mathbb{V}(\mathfrak{g}, \ell)$, along the map from $\overline{\mathrm{M}}_{g,1}$ to $\overline{\mathrm{M}}_g$ given by dropping the marked point. We then construct the fibers on $\overline{\mathrm{M}}_{g,1}$ with the zero weight. If C is a stable curve then one normalizes it at the nodes and uses "Factorization" to construct the fibers (see Section 7).

3.2 A geometric description of the fibers at smooth curves

This result can be said very generally, but for the purposes of the talk, and to avoid a lot of notation I will state it for bundles in type A on M_g:

Theorem 3.2. [BL94, Fal94, KNR94] *For $[C] \in \mathrm{M}_g$,*

$$\bigoplus_{m\in\mathbb{Z}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_{[C]} \cong \bigoplus_{m\in\mathbb{Z}} H^0(M_C(r), A_C(r)^{\ell m}),$$

where $M_C(r)$ is the moduli space of semi-stable vector bundles on C of rank r with trivializable determinant, and A is an ample line bundle on it.

So in particular,

$$\mathbf{Proj}\Big(\bigoplus_{m\in\mathbb{Z}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_{[C]}\Big) \cong M_C(r),$$

and one says the bundle $\mathbb{V}(\mathfrak{sl}_r, \ell)$ has a geometric interpretation at the point $[C] \in \mathrm{M}_g$. In fact, Laszlo and Sorger in [LS97], showed that such geometric interpretations for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ exist at points represented by stable pointed curves $(C; p_1, \ldots, p_n)$, as long as C is smooth.

Question 3.3. *Do such geometric interpretations for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ exist at points represented by stable pointed curves $(C; p_1, \ldots, p_n)$, where C has singularities?*

We showed in [BGK15] that the answer is no, not necessarily!

Theorem 3.4. [BGK15]*Let C be the singular curve of genus 2 with a single separating node. There is no polarized pair (X, A) such that*

$$\bigoplus_{m \in \mathbb{Z}} \mathbb{V}(\mathfrak{sl}_2, m)|^*_{[C]} \cong \bigoplus_{m \in \mathbb{Z}} H^0(X, A^m),$$

To show this we prove that if $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, 1)$ has geometric interpretation at all boundary points on $\overline{\mathrm{M}}_2$, then then

$$c_1(\mathbb{V}[m]) = \binom{m+3}{4} c_1(\mathbb{V}) = \frac{(m+3)(m+2)(m+1)m}{24} \cdot c_1(\mathbb{V}) \tag{5}$$

which we can show fails by intersecting with F-curves. There are two types of F-curves on $\overline{\mathcal{M}}_2$. The first is the image of a clutching map from $\overline{\mathcal{M}}_{0,4}$ for which points are identified in pairs. The second is the image of a map from $\overline{\mathcal{M}}_{1,1}$ given by attaching a point $(E, p) \in \mathcal{M}_{1,1}$, gluing the curves at the marked points. One obtains a contradiction when we intersect with either type of F-curve, even just at $m = 2$.

We do know that sometimes there are geometric interpretations. Here are two types of results along those lines:

Theorem 3.5. [BGK15]*Geometric interpretations hold at all points for all bundles on $\overline{\mathrm{M}}_{g,n}$ of rank one.*

While it is difficult to find bundles of rank one on $\overline{\mathrm{M}}_{g,n}$ for positive genus g, Theorem 3.5 implies that bundles $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, 1)$ on $\overline{\mathrm{M}}_{0,n}$ have geometric interpretations at all points $x \in \overline{\mathrm{M}}_{0,n}$.

Theorem 3.6. [BG16]*Given $[C] \in \overline{\mathcal{M}}_g$, and a positive integer r, there exists a projective polarized pair $(\mathcal{X}_C(r, \ell), \mathcal{L}_C(r, \ell))$, and a positive integer ℓ such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, m\ell)|^*_{[C]} \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathrm{H}^0(\mathcal{X}_C(r, \ell), \mathcal{L}_C(r, \ell)^{\otimes m}). \tag{6}$$

We can be more precise about ℓ in some cases:

1. *For general r if C has only nonseparating nodes, $\ell \geqslant 1$;*
2. *For $r = 2$, ℓ divisible by 2;*
3. *For general r, and C with separating nodes, we know such an ℓ exists.*

4 Open problems

In my talk I mentioned three open problems about the moduli space of curves and vector bundles of conformal blocks.

4.1 Global generation of vector bundles for regular VOAs

As I mentioned earlier, in [NT05] Nagatomo and Tsuchiya proved that the conformal blocks that come from regular chiral vertex operator algebras in [NT05], they satisfy "Factorization" and "Propagation of Vacua".

One can ask whether these vector bundles are globally generated on $\overline{\mathrm{M}}_{0,n}$.

4.2 The nef cone of $\overline{\mathrm{M}}_{0,n}$ and vector bundles of conformal blocks

Naturally one wonders how much of the nef cone is covered by first Chern classes of vector bundles of conformal blocks. If the whole cone is covered, then every nef divisor is semi-ample. But then one could hope that there would be an infinite number of extremal rays, contradicting the F-Conjecture.

If the cone is covered by conformal blocks and there are just finitely many extremal rays, then there must necessarily be a lot of identities between first Chern classes, or vanishing results.

As an example of a type of identity, we have

Proposition 4.1. [BGM16] *Suppose* $\operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\mu},\ell} = 1$ *and* $\operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\mu}+\vec{\nu},\ell+m} = \operatorname{rk} \mathbb{V}_{\mathfrak{g},\vec{\nu},m} = \delta$. *Then*

$$c_1(\mathbb{V}_{\mathfrak{g},\vec{\mu}+\vec{\nu},\ell+m}) = \delta c_1(\mathbb{V}_{\mathfrak{g},\vec{\mu},\ell}) + c_1(\mathbb{V}_{\mathfrak{g},\vec{\nu},m}).$$

4.3 The problem of nonvanishing

To describe the problem of non-vanishing, it helps first to know the vanishing results.

4.3.1 Critical level vanishing and identities

The critical level, first defined by Fakhrudin [Fak12] for $\mathfrak{sl}_2$, is defined only for $\mathfrak{g} = \mathfrak{sl}_{r+1}$, while a similar concept called the theta level is defined for general Lie algebras $\mathfrak{g}$ [BGM15, BGM16]. As I will explain, the Chern classes of bundles are trivial if ℓ is above the critical level. In terms of first Chern classes, it seems that very many conformal blocks divisors are extremal in the nef cone, and the number of curves they contract increases as the level increases with respect to the pair $(\mathfrak{g}, \vec{\lambda})$. Moreover, sets of nontrivial classes where the Lie algebra and the weights are fixed but the level varies, have been shown to have interesting properties. For example on $\overline{\mathrm{M}}_{0,n}$, where $n = 2(g+1)$ is even $\{c_1(\mathbb{V}(\mathfrak{sl}_2, \omega_1^n, \ell)) : 1 \leqslant \ell \leqslant g = cl((\mathfrak{sl}_2, \omega_1^n)\}$, forms a basis of $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$ [?ags].

Definition 4.2. *If $r+1$ divides $\sum_{i=1}^n |\lambda_i|$, we refer to*

$$cl(\mathfrak{sl}_{r+1}, \vec{\lambda}) = -1 + \frac{\sum_{i=1}^n |\lambda_i|}{r+1},$$

as the critical level for the pair $(\mathfrak{sl}_{r+1}, \vec{\lambda})$. If $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, and if $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, then $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ is called a critical level bundle, and $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ is called a critical level divisor.

Note that if $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, then $r = cl(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T)$, where $\vec{\lambda}^T = (\lambda_1^T, \ldots, \lambda_n^T)$. Here λ_i^T is the weight associated to the transpose of the Young diagram associated to the weight λ_i. In particular, $|\lambda_i| = |\lambda_i^T|$, and so

$$\sum_{i=1}^n |\lambda_i| = (r+1)(\ell+1) = (\ell+1)(r+1) = \sum_{i=1}^n |\lambda_i^T|.$$

In particular, critical level bundles come in pairs:

The following theorem was first proved by Fakhruddin for $\mathfrak{sl}_2$ in [Fak12]:

Theorem 4.3. [Fak12, BGM15] *If $\ell = cl(\mathfrak{sl}_{r+1}, \vec{\lambda})$, then*

1. *$c_k(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell + c)) = 0$, for $c \geqslant 1$; and*
2. *$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T, r))$.*

4.3.2 Examples

1. The bundle $\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)$ is at the critical level for $n = (r+1)(\ell+1)$. In [BGM15] we showed that the first Chern classes are all nonzero, and by Theorem 4.3, for $n = (r+1)(\ell+1)$,

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r)); \text{ and}$$

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell + c)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r + c)) = 0 \text{ for all } c \geqslant 1.$$

2. The bundle $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1 + \omega_3)^3\}, 3)$ is at the critical level, and its first Chern class is self dual.

Remark 4.4. *The main applications of vanishing above the critical level are extremality tests, and criteria for showing that maps given by conformal blocks divisors factor through contraction maps to Hassett spaces.*

The bundle $\mathbb{V}(\mathfrak{sl}_4, \{\omega_2 + \omega_3, \omega_1, \omega_1 + 2\omega_2, 2\omega_1 + \omega_3\}, 3)$ is at the critical level, (and it is below the theta level (which is 3.5)). The rank of $\mathbb{V}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3}$ on $\overline{\mathrm{M}}_{0,4}$ is one, while the dimension of the vector space of coinvariants $\mathbb{A}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}}$ is 2. A calculation shows that $\mathbb{D}_{\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3} = 0$.

Examples like this have led us in [BGM16] to ask when divisors are nonzero.

Question 4.5. *What are necessary and sufficient conditions for a triple $(\mathfrak{g}, \vec{\lambda}, \ell)$ that guarantee that the associated conformal blocks divisor $\mathbb{D}_{\mathfrak{g}, \vec{\lambda}, \ell}$ is nonzero?*

5 More on the moduli space

We start with the case where $n = 0$, and by looking at the boundary of the moduli space, see how stable pointed curves arise naturally.

5.1 $\overline{\mathrm{M}}_g$

Definition 5.1. *A **stable curve** is a complete connected curve with only nodes as singularities and only finitely many automorphisms.*

Remark 5.2. *In order for a curve to have a finite number of automorphisms, any rational component must meet any other component of the curve in at least three points.*

Definition 5.3. *For $g = \dim \mathrm{H}^1(C, \mathcal{O}_C) \geqslant 2$, consider the contravariant functor:*

$$\overline{\mathcal{M}}_g : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_g(\mathrm{T}),$$

where $\overline{\mathcal{M}}_g(\mathrm{T})$ is the set of flat proper morphisms $\pi : \mathcal{F} \to \mathrm{T}$ such that every fiber $\mathcal{F}_t$ is a stable curve of genus g modulo isomorphism over T.

Theorem 5.4. [DM69]*There exists a coarse moduli space $\overline{\mathrm{M}}_g$ for the moduli functor $\overline{\mathcal{M}}_g$; Moreover, $\overline{\mathrm{M}}_g$ is a projective variety that contains M_g as a dense open subset.*

Figure 1: Components of the boundary of $\overline{\mathrm{M}}_g$

5.2 The boundary of $\overline{\mathrm{M}}_g$

The boundary is a union of components:

$$\overline{\mathrm{M}}_g \backslash \mathrm{M}_g = \cup_{i=0}^{\lfloor \frac{g}{2} \rfloor} \Delta_i,$$

- Δ_0 is the closure of the locus of curves with a single non-separating node, and
- for $i > 0$, Δ_i is the closure of the locus of curves with a single separating node whose normalization consists of a curve of genus i and a curve of genus $g - i$.

5.3 $\overline{\mathrm{M}}_{g,n}$

As one can see in the images pictured in Figure 1, moduli of pointed curves come up naturally even if one is only interested in studying $\overline{\mathrm{M}}_g$: Each component of the boundary is the image of a morphism from a variety (or product of varieties) that (coarsely) represent a more general moduli functor

$$\overline{\mathrm{M}}_{g-1,2} \twoheadrightarrow \Delta_0, \quad \text{and for } 1 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor, \quad \overline{\mathrm{M}}_{i,1} \times \overline{\mathrm{M}}_{g-i,1} \twoheadrightarrow \Delta_i.$$

Definition 5.5. *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection $p_1, p_2, \ldots, p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only a finite number of automorphisms.*

Definition 5.6. *For $g = 0$, let $n \geqslant 3$, and for $g = 1$, let $n \geqslant 1$:*

$$\overline{\mathcal{M}}_{g,n} : (\mathcal{S}ch_{\mathrm{k}}) \to (\mathcal{S}ets), \quad \mathrm{T} \mapsto \overline{\mathcal{M}}_{g,n}(\mathrm{T}),$$

where $\overline{\mathcal{M}}_{g,n}(\mathrm{T})$ is the set of proper families $(\pi : \mathrm{X} \to \mathrm{T}; \{\sigma_i : T \to \mathrm{X}\}_{i=1}^n)$ such that the fiber $(\mathrm{X}_t, \{\sigma_i(t)\}_{i=1}^n)$, at every geometric point $t \in \mathrm{T}$ is a stable n-pointed curve of genus g modulo isomorphism over T.

Theorem 5.7. [KM76, Knu83a, Knu83b] *There exists a coarse moduli space $\overline{\mathrm{M}}_{g,n}$ for the moduli functor $\overline{\mathcal{M}}_{g,n}$; it is a projective variety that contains $\mathrm{M}_{g,n}$ as a dense open subset. Moreover, $\overline{\mathrm{M}}_{0,n}$ is a smooth projective variety that is a fine moduli space for $\overline{\mathcal{M}}_{0,n}$.*

When $g = 0$, the moduli space $\overline{\mathrm{M}}_{0,n}$ represents the functor $\overline{\mathcal{M}}_{0,n}$, and moreover it is a smooth projective rational variety. Kapranov showed how to construct $\overline{\mathrm{M}}_{0,n}$ as both a Chow and Hilbert quotient using Veronese curves, and alternatively as a Chow quotient using the Grassmannian $\mathrm{G}(2,n)$. Keel gave an alternative blowup construction allowing an explicit description for the Chow ring of $\overline{\mathrm{M}}_{0,n}$. A third blowup construction was given by Chen, Krashen and myself, which we generalized to a related moduli space $T_{d,n}$ parametrizing n-pointed rooted trees of projective spaces of dimension d ($T_{1,n} \cong \overline{\mathrm{M}}_{0,n+1}$).

6 Just enough about the affine Lie algebra $\hat{\mathfrak{g}}$

In Section 6.1 we will define the bracket that gives $\hat{\mathfrak{g}}$ the structure of a Lie algebra, and in Section 6.2 we'll define the $\hat{\mathfrak{g}}$-modules H_λ used in the definition of vector bundles of conformal blocks.

6.1 The bracket on $\hat{\mathfrak{g}}$

Let $\hat{\mathfrak{g}} = \big(\mathfrak{g} \otimes \mathbb{C}((\xi))\big) \oplus \mathbb{C} \cdot c$, where $\mathbb{C}((\xi))$ is the field of Laurant power series over $\mathbb{C}$ in 1 variable, and $c \in \mathfrak{g}$ is in the center of $\hat{\mathfrak{g}}$. To define the bracket for $\hat{\mathfrak{g}}$, we set

$$[\mathrm{X}\otimes f(\xi), \mathrm{Y}\otimes g(\xi)] = [\mathrm{X},\mathrm{Y}] \otimes f(\xi)g(\xi) + (\mathrm{X},\mathrm{Y}) \cdot \mathrm{Res}(g(\xi)df(\xi)) \cdot c,$$

where X, $\mathrm{Y} \in \mathfrak{g}$.

Typical elements in $\hat{\mathfrak{g}}$ are of the form $\sum_{i=1}^{n} \mathrm{X}_i \otimes f_i(\xi) \oplus \lambda c$, and $\sum_{j=1}^{n} \mathrm{Y}_j \otimes g_j(\xi) \oplus \mu c$, so using that $[c, \square] = [\square, c] = 0$, for all $\square \in \hat{\mathfrak{g}}$, since c is central in $\hat{\mathfrak{g}}$:

$$(7)\quad [\sum_{i=1}^{n} \mathrm{X}_i \otimes f_i(\xi) \oplus \lambda c, \sum_{j=1}^{n} \mathrm{Y}_j \otimes g_j(\xi) \oplus \mu c]$$
$$= [\sum_{i=1}^{n} \mathrm{X}_i \otimes f_i(\xi), \sum_{j=1}^{n} \mathrm{Y}_j \otimes g_j(\xi)] = \sum_{ij} [\mathrm{X}_i \otimes f_i(\xi), \mathrm{Y}_j \otimes g_j(\xi)].$$

So the upshot is that we really only need to know that the given definition for $[\mathrm{X}\otimes f(\xi), \mathrm{Y}\otimes g(\xi)]$ makes sense and is well defined. That is, we need to check anti-symmetry and the Jacobi identity.

Claim 6.1. *The proposed Lie bracket for $\hat{\mathfrak{g}}$ satisfies the Jacobi identity:*

$$\begin{aligned}(8)\quad & [[X\otimes f(\xi), Y\otimes g(\xi)], Z\otimes h(\xi)] \\ &\qquad = [X\otimes f(\xi),[Y\otimes g(\xi), Z\otimes h(\xi)]] - [Y\otimes g(\xi),[X\otimes f(\xi), Z\otimes h(\xi)]].\end{aligned}$$

Proof. Using a bit of shorthand, we drop the variable ξ writing $\overline{gf'}$ instead of $\mathrm{Res}(g(\xi)df(\xi))$, we can express the left hand side of the equation as:

$$[[X,Y]\otimes fg \oplus \big((X,Y)\overline{gf'}\big)c, Z\otimes h] = A + B, \tag{9}$$

where

$$A = [[X,Y],Z]\otimes fgh, \quad \text{and} \quad B = \big([X,Y],Z\big)\overline{h(fg)'}c.$$

The right hand side of the equation can be written as:

$$\begin{aligned}(10)\quad & [X\otimes f, [Y,Z]\otimes gh + (Y,Z)\overline{g'h}\cdot c] - [Y\otimes g, [X,Z]\otimes fh + (x,Z)\overline{hf'}c] \\ &\qquad = [X,[Y,Z]]\otimes fgh \oplus (X,[Y,Z])\cdot\overline{ghf'}\cdot c \\ &\qquad\qquad \ominus [Y,[X,Z]]\otimes gfh \oplus (Y,[X,Z])\cdot\overline{fhg'}\cdot c = A' + B',\end{aligned}$$

where

$$A' = [X,[Y,Z]]\otimes fgh, \quad \text{and} \quad B' = \Big(\big(X,[Y,Z]\big)\overline{ghf'} - \big(Y,[X,Z]\big)\overline{fhg'}\Big)c.$$

One has that $A = A'$ by the Jacobi identity for the Lie bracket for $\mathfrak{g}$, and so it remains to check that $B = B'$. Using the following three identities:

1. the product rule: $(fg)' = f'g + fg'$;
2. $\big([X,Y],Z\big) = \big(X,[Y,Z]\big)$ (Lemma 6.2); and
3. $\big([X,Y],Z\big) = -\big(Y,[X,Z]\big)$ (Lemma 6.3),

we write

$$\begin{aligned}(11)\quad B &= \big([X,Y],Z\big)\overline{h(fg)'}c \\ &= \big([X,Y],Z\big)\overline{hf'g}c + \big([X,Y],Z\big)\overline{hfg'}c \\ &= \big(X,[Y,Z]\big)\overline{hf'g}c - \big(Y,[X,Z]\big)\overline{hfg'}c = B'.\end{aligned}$$

∎

The following identity is referred to as the Frobeneous property of the Killing form.

Lemma 6.2. $\big([X,Y],Z\big) = \big(X,[Y,Z]\big)$

Proof. By definition of the Killing form, and using that Trace is invariant under cyclic permutations (so $\mathrm{Trace}(abc) = \mathrm{Trace}(cab)$), we write:

$$\begin{aligned}(12)\quad \big(X,[Y,Z]\big) &= \mathrm{Trace}\big(\mathrm{ad}(X)\,\mathrm{ad}(Y)\,\mathrm{ad}(Z)\big) - \mathrm{Trace}\big(\mathrm{ad}(X)\,\mathrm{ad}(Z)\,\mathrm{ad}(Y)\big)\\ &= \mathrm{Trace}\big(\mathrm{ad}(X)\,\mathrm{ad}(Y)\,\mathrm{ad}(Z)\big) - \mathrm{Trace}\big(\mathrm{ad}(Y)\,\mathrm{ad}(X)\,\mathrm{ad}(Z)\big)\\ &\qquad \mathrm{Trace}\big((\mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X)\,\mathrm{ad}(Z)\big) = \big([X,Y],Z\big).\end{aligned}$$

□

Lemma 6.3. $\big([X,Y],Z\big) = -\big(Y,[X,Z]\big)$

Proof. For the left hand side of the equation, using the symmetry of the Killing form:

$$\big([X,Y],Z\big) = \big(Z,[X,Y]\big).$$

For the right hand side, using that the Lie bracket is antisymmetric, while the Killing form is symmetric, we write:

$$-\big(Y,[X,Z]\big) = \big(Y,[Z,X]\big) = \big([Z,X],Y\big).$$

Now these are the same by Lemma 6.2:

$$\big([Z,X],Y\big) = \big(Z,[X,Y]\big).$$

□

Claim 6.4. $[X\otimes f(\xi), X\otimes f(\xi)] = 0$

Proof. Using that $[X,X] = 0$ since $\mathfrak{g}$ is a Lie algebra, and it's Lie bracket is of course anti-symmetric, and moreover, since $\frac{d}{d\xi}\frac{1}{2}f^2 = f(\xi)f'(\xi)d\xi$. So

$$\mathrm{Res}_{\xi=0}\,\Big(\frac{d}{d\xi}\frac{1}{2}f^2\Big) = \mathrm{Res}_{\xi=0}\,\big(f(\xi)f'(\xi)d\xi\big) = 0.$$

We can then write

$$[X\otimes f(\xi), X\otimes f(\xi)] = \big([X,X]\otimes f(\xi)\cdot f(\xi)\big) \oplus (x,x)\,\mathrm{Res}_{\xi=0}\,\big(f(\xi)f'(\xi)d\xi\big) = 0.$$

□

6.2 $\hat{\mathfrak{g}}$ modules M_λ and H_λ

There is a bijection between the intersection $\mathcal{W} \cap \Lambda_W$ of the Weyl chamber $\mathcal{W}$ and the weight lattice Λ_W and the set of irreducible representations for a given Lie algebra $\mathfrak{g}$. Given $\lambda \in \mathcal{W} \cap \Lambda_W$, there is a corresponding finite irreducible representation V_λ for $\mathfrak{g}$. In particular, V_λ is a $\mathfrak{g}$-module.

We are going to use V_λ to construct a representation M_λ for $\hat{\mathfrak{g}}$.

To construct M_λ, we use the following:

$$\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi, \quad \text{and} \quad \hat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[[\xi^{-1}]]\xi^{-1},$$

which we regard as Lie subalgebras of $\hat{\mathfrak{g}}$. One can show that

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c \oplus \hat{\mathfrak{g}}_-.$$

We'll also use the "positive" Lie sub-algebra

$$\hat{\mathfrak{p}}_+ = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c,$$

along with the universal enveloping algebras $\mathcal{U}(\hat{\mathfrak{g}})$ and $\mathcal{U}(\hat{\mathfrak{p}}_+)$.

Definition 6.5. $\mathrm{M}_\lambda := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{p}}_+)} V_\lambda$.

Remark 6.6. *Definition 6.5 makes sense: taking such a tensor product is legal:*

1. *If $\mathfrak{g}_1$ is any subalgebra of a Lie algebra $\mathfrak{g}_2$, then the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ extends to a monomorphism $\mathcal{U}(\mathfrak{g}_1) \hookrightarrow \mathcal{U}(\mathfrak{g}_2)$. Furthermore $\mathcal{U}(\mathfrak{g}_2)$ is a free $\mathcal{U}(\mathfrak{g}_1)$ module. So in particular, as $\hat{\mathfrak{p}}_+ \hookrightarrow \hat{\mathfrak{g}}$, we have that $\mathcal{U}(\hat{\mathfrak{g}})$ is a free $\mathcal{U}(\hat{\mathfrak{p}}_+)$ module.*

2. *V_λ is a $\hat{\mathfrak{p}}_+$-module. To see that this is true, note that since V_λ is a $\mathfrak{g}$-representation, there is a Lie algebra homomorphism*
$$\mathfrak{g} \to \mathrm{End}(V_\lambda).$$
Since $\hat{\mathfrak{p}}_+ = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c$, we can let $\hat{\mathfrak{g}}_+$ act by zero and $\mathbb{C} \cdot c$ act by taking
$$\mathbb{C} \cdot c \to \mathrm{End}(V_\lambda), \; \alpha c \mapsto [V_\lambda \to V_\lambda, \;\; v \mapsto (\alpha \ell) v],$$
where here ℓ is the level.

Claim 6.7. *M_λ is a representation for $\hat{\mathfrak{g}}$*

Proof. To show that there is a Lie algebra morphism

$$\hat{\mathfrak{g}} \to \mathrm{End}(\mathrm{M}_\lambda),$$

we may show there is a map of associative algebras

$$\mathcal{U}(\hat{\mathfrak{g}}) \to \mathrm{End}(\mathrm{M}_\lambda).$$

But by construction, $\mathcal{U}(\hat{\mathfrak{g}})$ acts on the left of M_λ, and so this is true. □

Definition 6.8. *We set* $\mathrm{H}_\lambda = \mathrm{M}_\lambda / \mathrm{I}_\lambda$.

Since $U(\hat{\mathfrak{g}})$ is isomorphic, as a $U(\hat{\mathfrak{g}}_-)$-module to $U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U(\mathfrak{p}_+)$:

$$\begin{aligned}(13)\quad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c &= \mathfrak{g} \otimes \left(\mathbb{C}[[\xi^{-1}]]\xi^{-1} \otimes_{\mathbb{C}} \mathbb{C}[[\xi]]\xi\right) \oplus \mathbb{C} \cdot c \\ &\cong \mathfrak{g} \otimes \mathbb{C}[[\xi^{-1}]]\xi^{-1} \otimes_{\mathbb{C}} \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi \oplus \mathbb{C} \cdot c \cong \hat{\mathfrak{g}}_- \otimes_{\mathbb{C}} \hat{\mathfrak{p}}_+.\end{aligned}$$

So we can rewrite the module M_λ as:

$$\mathrm{M}_\lambda \cong U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda \cong U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U(\hat{\mathfrak{p}}_+) \otimes_{U(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda \cong U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} \mathrm{V}_\lambda\,.$$

In particular, elements in $\mathrm{M}_\lambda := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{p}}_+)} \mathrm{V}_\lambda$ look like elements $v \in \mathrm{V}_\lambda$ times all the negative stuff in $\hat{\mathfrak{g}}$.

With the notation above, M_λ contains a unique (see eg [TUY89, Bea96]) maximal proper submodule I_λ generated by an element

$$J_\lambda = (\mathrm{X}_\theta \otimes \xi_i^{-1})^{\ell - (\theta,\lambda)+1} \otimes v_\lambda \in \mathrm{M}_\lambda, \quad \text{and} \quad \mathrm{I}_\lambda = U(\hat{\mathfrak{g}}_-) J_\lambda,$$

where here θ is the longest root, $\mathrm{X}_\theta \in \mathfrak{g}$ is the corresponding coroot, and v_λ is the highest weight vector associated to λ. We set

$$\mathrm{H}_\lambda = \mathrm{M}_\lambda / \mathrm{I}_\lambda\,.$$

We see that H_λ is a $(\mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c)$-module. The subspace of H_λ annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic as a $\mathfrak{g}$-module to V_λ. So we identify V_λ with this subspace of H_λ annihilated by $\hat{\mathfrak{g}}_+$.

7 Factorization, propagation of vacua, and Beauville's quotient construction

Theorem 7.4, originally proved by Tsuchiya, Ueno and Yamada [TUY89, Prop 2.2.6], explains how a vector bundle of conformal blocks at a point on the moduli space where the underlying curve has a node, factors into sums and products of bundles on the normalization of the curve where the sum is taken over all possible weights at points over which the normalization is "glued" to make the original curve. Applications of Factorization include inductive formulas for the rank and Chern classes of the bundle.

These notes closely follow [TUY89, Prop 2.2.6], and [Bea96]. Examples from [BGM16] are given.

7.1 Factorization

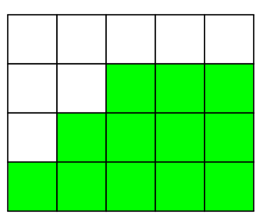
Definition 7.1. *Given a weight $\mu \in \mathcal{P}_\ell(\mathfrak{g})$, let $\mu^\star \in \mathcal{P}_\ell(\mathfrak{g})$ be the element with the property that $-\mu^\star$ is the lowest weight of the weight space V_μ.*

Example 7.2. *If $\mu \in \mathcal{P}_\ell(\mathfrak{sl}_2)$, then $\mu^\star = \mu$.*

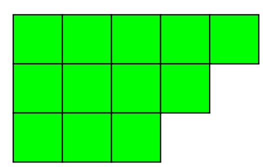
Example 7.3. *For $\mathfrak{g} = \mathfrak{sl}_{r+1}$ we express a weight λ_i as a sum $\lambda_i = \sum_{j=1}^{r} c_j \omega_j$, and λ_i has a corresponding Young diagram that fits into an $(r+1) \times \ell$ sized grid, where since λ_i is normalized, the last row is empty. In terms of Young diagrams, the level is the number of "filled in" boxes across the top, and $|\lambda_i|$ means the total number of boxes "filled in" altogether. To find the Young diagram corresponding to $\lambda^\star$ we fill in the boxes in the diagram directly below the boxes corresponding to λ, and then rotate by 180 degrees to get the Young diagram associated to the weight λ^*. For example, if $r+1 = 4$, and $\ell \geqslant 5$ for the weight λ pictured in white on the left below, then the dual weight $\lambda^\star$ is pictured in green on the right.*

Figure 2

$\lambda = 3\omega_1 + \omega_2 + \omega_3$
for $\mathfrak{sl}_4$,
and level $\ell(\lambda) \geqslant 5$.



$\lambda^\star = \omega_1 + \omega_2 + 3\omega_3$.



Theorem 7.4 (Factorization). *Let $(C_0; p_1, \ldots, p_n)$ be a stable n-pointed curve of genus g where C_0 has a node x_0.*

1. *If x_o is a non-separating node, $\nu : C \to C_0$ the normalization of C_0 at x_0, and $\nu^{-1}(x_0) = \{x_1, x_2\}$, then*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \mu \cup \mu^\star\}, \ell)_{(C;\vec{p} \cup \{x_1,x_2\})}.$$

2. *If x_0 is a separating node, $\nu : C_1 \cup C_2 \to C_0$ the normalization of C_0 at x_0 and $\nu^{-1}(x_0) = \{x_1, x_2\}$, with $x_i \in C_i$, then*

$$\begin{aligned} &\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \\ &\cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \lambda(C_1) \cup \{\mu\}, \ell)|_{(C_1;\{p_i \in C_1\} \cup \{x_1\})} \otimes \mathbb{V}(\mathfrak{g}, \lambda(C_2) \cup \{\mu^\star\}, \ell)|_{(C_2;\{p_i \in C_2\} \cup \{x_2\})}, \end{aligned} \tag{14}$$

   where $\lambda(C_i) = \{\lambda_j | p_j \in C_i\}$.

Definition 7.5. *The weights μ and $\mu^* \in \mathcal{P}_\ell(\mathfrak{g})$ that occur in Theorem 7.4 are called the restriction data for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ at the point $(C_0; \vec{p})$.*

Example 7.6. [BGM16] *We will factorize the bundle $\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)$ on $\overline{\mathcal{M}}_{0,4}$ at the two types of points $(C; p_1, \ldots, p_4)$, where the curve C has one node: the first type $X_1 = (C_{11} \cup C_{12}; p_1, \ldots, p_4)$ where C_{11} is labeled by p_1 and p_2 and C_{12} by p_3 and p_4; and the second type of curve $X_2 = (C_{21} \cup C_{22}; p_1, \ldots, p_4)$ where C_{21} is labeled by p_1 and p_3 and C_{22} by p_3 and p_4.*

1. *If $r + 1 = 2$ this is $\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_1\}, \ell)$, and we obtain:*

$$\begin{aligned} &\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_1} \\ &\quad \cong \bigoplus_{\substack{m \geqslant 0 \\ even}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, m\omega_1\}, \ell)|_{(C_{11},p_1,p_2,x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\ell\omega_1, \ell\omega_r, m\omega_1\}, \ell)|_{(C_{12},p_3,p_4,x_2)}. \end{aligned}$$

   As we'll see later, the only term in the sum above that gives bundles of nonzero rank occurs when $m = 0$, and that both bundles have rank one.

$$\begin{aligned} &\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_2} \\ &\cong \bigoplus_{\substack{m \geqslant 0 \\ m+\ell \equiv 1 (\mathrm{mod}\, 2)}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{21},p_1,p_3,x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{22},p_2,p_4,x_2)}. \end{aligned}$$

   Again, we'll see that the only term above that gives two bundles of nonzero rank occurs when $m = (\ell - 1)$, and has rank one in this case.

2. *If $r+1=3$ this is $\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\omega_1,(\ell-1)\omega_1+\omega_2,\ell\omega_1\},\ell)$, and we obtain, for*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\omega_1,(\ell-1)\omega_1+\omega_2,\ell\omega_1\},\ell)|_{X_1}\\ \cong &\bigoplus_{\substack{\mu=c_1\omega_1+c_2\omega_2\\ c_1+2c_2\equiv 1(\mathrm{mod}\,3)}}\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\omega_1,\mu\},\ell)|_{(C_{11},p_1,p_2,x_1)}\otimes\mathbb{V}(\mathfrak{sl}_3,\{\ell\omega_1,\ell\omega_r,\mu^\star\},\ell)|_{(C_{12},p_3,p_4,x_2)}.\end{aligned}$$

We'll later see that the only summand on the right hand side with nonzero rank is the one with $\mu=\omega_1$ (so $c_1=1$, and $c_2=0$).

$$\begin{aligned}(15)\quad &\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\omega_1,(\ell-1)\omega_1+\omega_2,\ell\omega_1\},\ell)|_{X_2}\\ \cong &\bigoplus_{\substack{\mu=c_1\omega_1+c_2\omega_2\\ \ell+c_1+2c_2\equiv 1(\mathrm{mod}\,3)}}\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\ell\omega_1,\mu\},\ell)|_{(C_{21},p_1,p_3,x_1)}\otimes\mathbb{V}(\mathfrak{sl}_3,\{\omega_1,\ell\omega_2,\mu^\star\},\ell)|_{(C_{22},p_2,p_4,x_2)}.\end{aligned}$$

We'll later see that the only summand on the right hand side with nonzero rank is the one with $\mu=(\ell-1)\omega_2$ (so $c_1=0$, and $c_2=(\ell-1)$).

3. *In general:*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,\omega_1,(\ell-1)\omega_1+\omega_r,\ell\omega_r\},\ell)|_{X_1}\\ \cong &\bigoplus_{\substack{\mu=\sum_{i=1}^r c_i\omega_i\\ \sum_{i=1}^r i\cdot c_i+2\equiv 0(\mathrm{mod}(r+1))}}\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,\omega_1,\mu\},\ell)|_{(C_{11},p_1,p_2,x_1)}\otimes\mathbb{V}(\mathfrak{sl}_{r+1},\{(\ell-1)\omega_1+\omega_r,\ell\omega_r,\mu^\star,\ell).\end{aligned}$$

Moreover, one can show that the only summand on the right hand side with nonzero rank is the one with $\mu=\omega_{r-1}$.

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,\omega_1,(\ell-1)\omega_1+\omega_r,\ell\omega_r\},\ell)|_{X_2}\\ \cong &\bigoplus_{I}\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,(\ell-1)\omega_1+\omega_r,\mu\},\ell)|_{(C_{21},p_1,p_3,x_1)}\otimes\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,\ell\omega_r,\mu^\star,\ell)|_{(C_{22},p_2,p_4,x_2)},\end{aligned}$$

where we sum over the set

$$I=\{\mu=\sum_{i=1}^r c_i\omega_i\in\mathcal{P}_\ell(\mathfrak{sl}_{r+1}):\ \sum_{i=1}^r i\cdot c_i+\ell+r\equiv 0(\mathrm{mod}(r+1))\}.$$

We will eventually show that the only summand on the right hand side with nonzero rank is the one with $\mu=(\ell-1)\omega_r$ and $\mu^\star=(\ell-1)\omega_1$. We'll see that:

$$\mathrm{rk}\,\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,(\ell-1)\omega_1+\omega_r,(\ell-1)\omega_r\},\ell)=\mathrm{rk}\,\mathbb{V}(\mathfrak{sl}_{r+1},\{\omega_1,\ell\omega_r,(\ell-1)\omega_1,\ell)=1.$$

Remark 7.7. *This example exhibits the potential for the use of factorization to compute ranks, which is the idea behind the proof of the Verlinde formula. The comments made also indicate that there is a lot of vanishing happening – which is a foreshadowing of one of the open problems in the subject: that is to determine given $\mathfrak{g}$ and ℓ necessary and sufficient conditions which will guarantee that the first Chern class of the bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ is not zero. One indication is that it's rank is nonzero, which is actually enough for $\mathfrak{sl}_2$, but this is not in general. For example, while the rank of $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3\}, 3)$ is one, the first Chern class of this bundle is zero* [BGM16]. *We'll discuss this problem.*

7.2 Theorem 7.8

7.2.1 Notation

Let C be a possibly nodal curve, p_1, p_2, …, $p_s \in C$ be s smooth points of C, $U = C \backslash \{p_1, \ldots, p_s\}$ and let ξ_i be a local parameter of C near p_i. Then as before, for $\hat{\mathfrak{g}}(U) = \mathfrak{g} \otimes \mathcal{O}_C(U)$, we have an embedding

$$\hat{\mathfrak{g}}(U) \hookrightarrow \bigotimes_{i=1}^{s} \big(\mathfrak{g} \otimes \mathrm{k}((\xi_i))\big) \oplus \mathrm{k}\, c = \hat{\mathfrak{g}}_s, \ \ (\mathrm{X} \otimes f) \mapsto (\mathrm{X} \otimes f_{p_1}(\xi_1), \ldots, \mathrm{X} \otimes f_{p_n}(\xi_n), 0).$$

Given weights λ_1, …, $\lambda_s \in \mathcal{P}_\ell(\mathfrak{g})$, we have the $(\mathfrak{g} \otimes \mathrm{k}((\xi_i)) \oplus \mathrm{k}\, c)$-modules H_{λ_i}. The image of $\hat{\mathfrak{g}}(U)$ acts on $\mathrm{H}_{\vec{\lambda}} = \mathrm{H}_{\lambda_1} \otimes \cdots \otimes \mathrm{H}_{\lambda_s}$:

$$\hat{\mathfrak{g}}(U) \times \mathrm{H}_{\vec{\lambda}} \to \mathrm{H}_{\vec{\lambda}}, \ \ ((\mathrm{X} \otimes f), (w_1 \otimes \cdots \otimes w_s)) \mapsto \sum_{i=1}^{s} w_1 \otimes \cdots w_{i-1} \otimes (\mathrm{X} \otimes f_{p_i}) \cdot w_i \otimes \cdots \otimes w_s.$$

Now given any weight $\mu \in \mathcal{P}_\ell(\mathfrak{g})$, recall that the subspace of H_μ annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic as a $\mathfrak{g}$-module to V_μ, and so V_μ is identified with this subspace of H_μ. Given t points $q_1, \ldots, q_t \in U$, and weights, μ_1, μ_2, …, $\mu_t \in \mathcal{P}_\ell(\mathfrak{g})$ one can define an action of $\hat{\mathfrak{g}}(U)$ on $\mathrm{V}_{\vec{\mu}} = \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_t}$ by evaluation:

$$\hat{\mathfrak{g}}(U) \times \mathrm{V}_{\vec{\mu}} \to \mathrm{V}_{\vec{\mu}}, \ \ ((\mathrm{X} \otimes f), (v_1 \otimes \cdots \otimes v_t)) \mapsto \sum_{j=1}^{t} v_1 \otimes \cdots \otimes v_{j-1} \otimes (\mathrm{X} \otimes f(q_j)) \cdot v_j \otimes \cdots \otimes v_t.$$

Theorem 7.8. *With notation as above, the inclusions $\mathrm{V}_{\mu_j} \hookrightarrow \mathrm{H}_{\mu_j}$ induce an isomorphism*

$$[\mathrm{H}_{\vec{\lambda}} \otimes \mathrm{V}_{\vec{\mu}}]_{\hat{\mathfrak{g}}(U)} \xrightarrow{\sim} [\mathrm{H}_{\vec{\lambda}} \otimes \mathrm{H}_{\vec{\mu}}]_{\hat{\mathfrak{g}}(U \backslash \vec{q})} \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \vec{\mu}, \ell)|_{(C, \vec{p} \cup \vec{q})}.$$

Propagation of Vacua is a corollary of this.

7.3 Propagation of Vacua

Corollary 7.9. *[Propagation of Vacua] Let $q \in C \backslash \vec{p}$. There is a canonical isomorphism*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \{0\}, \ell)|_{(C,\vec{p}\cup\{q\})}.$$

Proof. (of Corollary 7.9) Apply Theorem 7.8 with $\{q_1, \ldots, q_t\} = \{q\}$, and $\{\mu_1, \ldots, \mu_t\} = \{\mu = 0\}$, using that $V_0 = 0$.

□

References

[Bea96] Arnaud Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 75–96. ↑3, 6.2, 7

[BL94] Arnaud Beauville and Yves Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419. ↑3.2

[BGM15] Prakash Belkale, Angela Gibney, and Swarnava Mukhopadhyay, *Vanishing and identities of conformal blocks divisors*, Algebr. Geom. **2** (2015), no. 1, 62–90. ↑4.3.1, 4.3, 1

[BGM16] P. Belkale, A. Gibney, and S. Mukhopadhyay, *Nonvanishing of conformal blocks divisors on $\overline{M}_{0,n}$*, Transform. Groups **21** (2016), no. 2, 329–353. ↑4.1, 4.3.1, 4.3.2, 7, 7.6, 7.7

[BGK15] Prakash Belkale, Angela Gibney, and Anna Kazanova, *Scaling of conformal blocks and generalized theta functions over $\overline{\mathcal{M}}_{g,n}$*, arXiv:1412.7204 [math.AG] (2015). ↑3.2, 3.4, 3.5

[BG16] Prakash Belkale and Angela Gibney, *On finite generation of the section ring of the determinant of cohomology line bundle*, arXiv: [math.AG] (2016). ↑3.6

[BzF01] David Ben-zvi and Edward Frenkel, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2001. ↑3

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109. ↑5.4

[Fak12] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176. ↑3, 4.3.1, 4.3.1, 4.3

[Fal94] Gerd Faltings, *A proof for the Verlinde formula*, J. Algebraic Geom. **3** (1994), no. 2, 347–374. ↑3.2

[GKM02] Angela Gibney, Sean Keel, and Ian Morrison, *Towards the ample cone of $\overline{M}_{g,n}$*, J. Amer. Math. Soc. **15** (2002), no. 2, 273–294. ↑2.3, 2

[Gib09] Angela Gibney, *Numerical criteria for divisors on $\overline{M}_g$ to be ample*, Compos. Math. **145** (2009), no. 5, 1227–1248. ↑2, 2

[KM13] Seán Keel and James McKernan, *Contractible extremal rays on $\overline{M}_{0,n}$*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 115–130. ↑2, 2

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55. ↑5.7

[Knu83a] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199. ↑5.7

[Knu83b] ———, *The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic* 0, Math. Scand. **52** (1983), no. 2, 200–212. ↑5.7

[KNR94] Shrawan Kumar, M. S. Narasimhan, and A. Ramanathan, *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. **300** (1994), no. 1, 41–75. ↑3.2

[LS97] Yves Laszlo and Christoph Sorger, *The line bundles on the moduli of parabolic G-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 4, 499–525. ↑3.2

[NT05] Kiyokazu Nagatomo and Akihiro Tsuchiya, *Conformal field theories associated to regular chiral vertex operator algebras, I: Theories over the projective line*, Duke Mathematical Journal **128** (2005), 393–471. ↑3, 4.1

[TUY89] Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, pp. 459–566. ↑3, 6.2, 7