GALOIS GROUPS IN ENUMERATIVE GEOMETRY AND APPLICATIONS

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Abstract. In 1870 Jordan explained how Galois theory can be applied to problems from enumerative geometry, with the group encoding intrinsic structure of the problem. Earlier Hermite showed the equivalence of Galois groups with geometric monodromy groups, and in 1979 Harris initiated the modern study of Galois groups of enumerative problems. He posited that a Galois group should be 'as large as possible in that it will be the largest group preserving internal symmetry in the geometric problem.

I will describe this background and discuss some work in a long-term project to compute, study, and use Galois groups of geometric problems, including those that arise in applications of algebraic geometry. A main focus is to understand Galois groups in the Schubert calculus, a well-understood class of geometric problems that has long served as a laboratory for testing new ideas in enumerative geometry.

1. Introduction

Cayley and Salmon studied smooth cubic surfaces $F = 0$ in $\mathbb{P}^3$ and showed that they contain exactly 27 lines. Moreover, these lines appear in a remarkable configuration. More precisely, suppose $F \in \mathbb{Q}[x]$ and let K be the field of definition of the 27 lines (typically $[K\colon \mathbb{Q}] = 27$). Denote by $\tilde{K}$ the Galois closure of $K/\mathbb{Q}$. Then $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ acts on the cubic surface F. Since it preserves the surface, it also preserves the 27 lines by permuting them. (Moreover, it preserves the configuration). And $\mathrm{Gal}(\tilde{K}/\mathbb{Q}) \subseteq E_6$ (here we denote by E_6 the Weyl group of the configuration of the 27 lines). Usually it is exactly E_6.

2. Modern View

Let $\mathbb{P}^{19}$ be the space of homogeneous cubics, let G be the Grassmannian of lines in $\mathbb{P}^3$, and let $L = \{(F, \ell) \subseteq \mathbb{P}^{19} \times \mathbb{C} \,:\, F|_\ell = 0\}$. Let $\pi\colon L \to \mathbb{P}^{19}$ be the projection map onto the first component. Then π is dominant and generically finite, so we can study the field rational functions on L modulo rational functions on $\mathbb{P}^{19}$. It has degree 27 and Galois group E_6.

Definition 2.1. A *branched cover* is a map $\pi : X \to B$, where $\dim X = \dim B$, $\overline{\pi(X)} = B$, and X is irreducible.

The map $\pi\colon L \to \mathbb{P}^{19}$ is a branched cover. Given a branched cover $\pi\colon X \to B$, let $\Sigma \subseteq B$ be the *discriminant locus*, i.e. the set of points where the fiber has degree different from $\deg \pi$. Then the map $\pi\colon X \setminus \pi^{-1}(\Sigma) \to B \setminus \Sigma$ is unramified over $\mathbb{C}$. In other words, generically, we can view X as a covering space of B in the topological sense.

Then the geometric monodromy group is the group of deck transformations. Earlier Hermite found that the monodromy group is the same as the Galois group. In 1979 Harris initiated the modern study of Galois groups.

Definition 2.2. A geometric problem (or branch covering or family of polynomial equations) is *deficient* if its monodromy/Galois group is not the full symmetric group.

In 1980, Harris observed that such Galois groups are "as large as possible" in that there is some structure in the problem constraining the group, but otherwise it is as large as possible. In the case of the 27 lines, it is the special configuration, which constrains the Galois group to be no larger than E_6, and in fact, it is E_6, so it is as large as possible given the structure.

In other words, if a Galois group is deficient, there should be some clear geometric structure explaining the deficiency. In Schubert calculus and algebraic statistics there were examples where it was observed that the Galois group was deficient before people knew what the structure was, so the goal became to figure out the geometric reason for the deficiency.

We will illustrate this phenomenon with a simple example. Suppose there is some involution acting which preserves the problems in your family. Eg. suppose you can parameterize your set by square matrices, and if you have a solution with a given matrix, then the transpose is also a solution. Assuming certain irreducibility conditions, the number of solutions is even, say $2n$. Then the Galois group is contained in $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. This is the group of signed permutations, denoted B_n. Often $G = B_n$, but sometimes $G = D_n$, which is the index 2 subgroup of alternating permutations.

3. Schubert calculus and results

The classical Schubert calculus is a well-understood class of geometric problems that has long served as a laboratory for testing new ideas in enumerative geometry. Generally speaking, it is all problems of determining the linear space of some type that have fixed position with respect to other general linear spaces.

An example of a classical problem in Schubert calculus is "What are the lines in $\mathbb{P}^3$ that meet 4 fixed lines?"

It is known that any three mutually skew lines determine a unique quadric (a hyperboloid of one sheet) on which they lie. Hyperboloids are double ruled, and there is one ruling in which these three lines lie, and there is another ruling which are lines that meet these. The lines in the second ruling meet all three of them. If I have a 4th line, this is a quadric, so it will meet the hyperboloid in 2 points, and if there's going to be a line that meets all 4 of them, it had better go through these points because the lines that meet the other three are on the hyperboloid. Furthermore, for every point on the hyperboloid there is a unique line in the second ruling that goes through that point, and that line meets the other three lines. And here there are 2 lines that meet all 4.

There are two ways to see that the Galois group is S_4: one is that it has to be a transitive permutation group on 2 elements, the other is by rotating the fourth line so that the two intersection points switch like a transposition in S_4.

This is boring from the point of view of deficiency. The Schubert calculus of enumerative geometry of this sort is more than just 4 lines in 3-space. There are a few things about it. We know every single Schubert problem. We know what they are, we know how many solutions it has, we know equations to them, we know everything about them. It's completely known, it's like if we classified all insects, we sort of know all of them. But because of this, it provides a laboratory for studying interesting phenomena. You can explore things in Schubert calculus, much in the same way in which people view toric varieties as a laboratory for studying phenomena in algebraic geometry. Similarly, I think Schubert calculus gives a way to do that for enumerative geometry questions.

The first example of a deficient Galois group of a Schubert problem was something that Vakil observed and Derksen generalized. Actually, at the same time, my research group found that we had found some deficient Schubert problems as well.

Vakil and Derksen have found infinite families of deficient Schubert problems. In fact, there are some in every Grassmannian group $\mathrm{Gr}(k,n)$ of k-planes in n-space, where $k, n-k \geq 4$.

When these examples of Vakil and Derksen came up, this laid to rest the idea we had from Harris's paper that this is just a low-dimensional, sporadic phenomenon.

What we found is that every deficient group is either an S_n acting on certain ordered set partitions of $\{1,2,...,n\}$, or iterated wreath products of these. For example, the symmetry group of a square can be thought of as a wreath product of the two diagonals, and it's $(S_2 \times S_2) \rtimes S_2$.

In all known examples we have found in Schubert calculus, either the Galois group is the full symmetric group or imprimitive.

Definition 3.1. A group $G \subseteq S_k$ is *primitive* if it preserves no partition of $\{1,2,...,k\}$. Otherwise, it is called imprimitive.

Definition 3.2. A group G is *n-transitive* if it sends every ordered n-tuple to any other ordered n-tuple.

For example, S_n is n-transitive, and A_n is $(n-2)$-transitive, but there are very few highly transitive groups. Furthermore, a 2-transitive group is primitive and so one which is imprimitive is therefore not 2-transitive.

Vakil gives a combinatorial/geometric method that can show that a Galois group contains the alternating group. It's a recursive method. This is based upon the idea of special position in enumerative geometry. A classical way you would solve a problem in enumerative geometry is you would take your conditions and you would put them in some special position. Then you claim that they're actually general, but because of their special position your problem breaks up into two smaller subproblems, and then you can apply recursion.

Using this, we've found that all Schubert problems in the Grassmannian $\mathrm{Gr}(2,n)$ are at least alternating, meaning that using this method, one can show that they contain the alternating group. We believe that they contain the full symmetric group, but that's the strength of our method (joint with Brooks, Martin del Campo). Another example is Schubert problems on $\mathrm{Gr}(3,n)$ are all 2-transitive.

In between transitive and 2-transitive is primitive, and the number of k for which there exists a primitive permutation group (once you throw away some obvious examples) is not dense in the integers, so primitive permutation groups are very rare.

In the special Schubert problems $\{\mathbb{H}_k \cap F_{n-k-a} \geq 1, \text{ for } n,k,a \in \mathbb{Z}\}$, the Galois groups are all 2-transitive (joint with White). Here the linear space of some type is $\mathbb{H}_k$ and other general linear space is F_{n-k-a}.

In the simple Schubert problems, all except two conditions are codimension 1, meaning $\dim(\mathbb{H}_k \cap F_{n-k}) \geq 1$, these are all at least alternating (joint with White and Williams)

Because of these results and examples, we know that the first interesting examples happen in the Grassmannian of 4-planes in 8-space, and so we studied Gr(4,8) (resp. Gr(4,9)) and we found 3,501 (resp. 36,767) Schubert problems that cannot be reduced to ones on a smaller Grassmannian. Most are at least alternating, except for 14 (resp. 148). (Note: the percentage is about the same in both). If we examine these, these fall into 3 (resp. 13) families.