ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2016

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 17th, 2016.

(1) (a) Determine, with proof, the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-5})$.

(b) Is an integrally closed ring a unique factorization domain? If so, prove it; if not, give and explain a counterexample.

(2) (a) Determine, with proof, the Jacobson radical, and the nilradical of $\mathbb{Q}[x]/(x^3)$.

(b) Show that an Artinian ring R contains only finitely many prime ideals.

(3) Let $p(x)$ and $q(x)$ be monic polynomials in $\mathbb{C}[x]$ that have the same set of zeros and such that $p(x)$ divides $q(x)$. Let $m = \deg(q)$. Prove that there exists a linear operator $T\colon \mathbb{C}^m \to \mathbb{C}^m$ such that the characteristic polynomial of T is q and the minimal polynomial of T is p.

(4) Let F be a finite field with 7 elements, and let β be a cubed root of 2, i.e. $\beta^3 = 2$.

(a) Compute the minimal polynomial for $\alpha = \beta + 1$ over F.

(b) Determine, with proof, whether or not $F(\alpha)$ is Galois over F.

(5) Let G be a finite group of order n, and $C_1, \ldots, C_k$ the orbits of G acting on itself by conjugation. The class equation for G is

$$n = |C_1| + |C_2| + \cdots + |C_k|.$$

(a) Write down the class equation for the dihedral group D_{20} of order 20.

(b) Can the equation

$$20 = 1 + 2 + 3 + 4 + 10$$

be the class equation for a group of order 20? If so, give an example; if not, give a proof.

(6) Let K be a finite field with 64 elements. Describe all of the subfields of K.