ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2017

Instructions:

• You have four hours to complete this exam. Attempt all six problems.
• The use of books, notes, calculators, or other aids is not permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.

Date: January 10, 2017.
(1) Consider the following matrices

\[ M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \]

Are the \( \mathbb{Z} \)-modules \( R_1 = \mathbb{Z}^2 / M_1 \cdot \mathbb{Z}^2 \) and \( R_2 = \mathbb{Z}^2 / M_2 \cdot \mathbb{Z}^2 \) isomorphic? If so, explain why and describe an isomorphism. If not, explain why not.

(2) Let \( K/F \) be a field extension, and \( \alpha \in K \) an element which is algebraic and of odd degree over \( F \). Show that \( F(\alpha^2) = F(\alpha) \).

(3) Give an example of a pair of square matrices \( A, B \) with \( \mathbb{C} \)-coefficients such that the minimal polynomials \( m_A(x) \) and \( m_B(x) \) are equal, the characteristic polynomials \( c_A(x) \) and \( c_B(x) \) are equal, but the matrices \( A \) and \( B \) are not conjugate.

(4) Let \( R \) be a ring with a unique prime ideal and \( \text{nil}(R) = (0) \). Prove that \( R \) is a field. [Here \( \text{nil}(R) \) stands for the nilradical of \( R \).

(5) Let \( p \in \mathbb{Z}_{>0} \) be a prime and \( R = \mathbb{F}_p[x] \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x] \).

(a) Prove that \( x \otimes 1 - 1 \otimes x \in \text{nil}(R) \).

(b) Prove that \( \text{nil}(R) = (x \otimes 1 - 1 \otimes x) \). [Hint: Consider the ring \( R/(x \otimes 1 - 1 \otimes x) \).

(6) Let \( \zeta \) be a primitive 7\(^{th}\) root of unity, considered as a complex number, and let \( F = \mathbb{Q}(\zeta) \) be the extension of the rational numbers obtained by adjoining \( \zeta \). Find, with proof, a primitive element that generates a subfield of \( F \) of degree 2 over \( \mathbb{Q} \).