1. Suppose that \( f_n : \mathbb{R} \to \mathbb{R} \) is a differentiable function for \( n = 1, 2, \ldots \), that
\[
M = \sup_{n,x} |f'_n(x)| < \infty,
\]
and that \( f(x) = \lim_{n \to \infty} f_n(x) \in \mathbb{R} \) exists for all \( x \in \mathbb{R} \).
(a) Is \( f \) continuous on \( \mathbb{R} \)? Prove or find a counterexample.
(b) Is \( f \) differentiable on \( \mathbb{R} \)? Prove or find a counterexample.
(c) Does \( \int_0^1 f(x) \, dx = \lim_{n \to \infty} \int_0^1 f_n(x) \, dx \)? Prove or find a counterexample.

2. How many zeros does \( f(z) = z^4 - 6z + 3 \) have in the annulus \( \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \)?

3. Suppose \( f \) is analytic on \( \mathbb{D} \setminus \{0\} \) and, for some \( \epsilon > 0 \), \( \text{Re} \, f(z) > 0 \) whenever \( 0 < |z| < \epsilon \). Prove that \( f \) has a removable singularity at \( 0 \).

4. Prove that for every \( f \in C([0,1]) \),
\[
\lim_{a \to \infty} a \int_0^1 e^{-ax} f(x) \, dx = f(0).
\]
Hint: compute the limit for \( f(x) = x^k, k = 0, 1, 2, \ldots \)

5. Compute for \( a \in (0,1) \) the value of the integral
\[
\int_0^\infty \frac{t^a}{1 + t^2} \, dt.
\]
Hint: consider an analytic function on \( \mathbb{C} \setminus [0, \infty) \) or a similar region.

6. (a) If \( f_n \to f \) in \( L^1([0,1], dx) \), prove that
\[
\int_0^1 e^f \, dx \leq \lim \inf_{n \to \infty} \int_0^1 e^{f_n} \, dx
\]
(b) Prove that strict inequality is possible.