1. Determine whether the following limit exists and, if so, find its value:

\[
\lim_{n \to \infty} \int_0^{1/n} \frac{n \, dt}{(1 + n^2 t^2)(1 + t^2)}
\]

2. Does there exist a nonconstant holomorphic function \(f : \mathbb{C} \to \mathbb{C}\), satisfying

\[
\lim_{|z| \to \infty} |z|^{-1/2} |f(z)| = 0 \quad \text{? If so, give an example. If not, explain why not.}
\]

3. Prove that, for all \(f \in C([0,1])\),

\[
\lim_{n \to \infty} n \int_0^1 x^n f(x) \, dx = f(1).
\]

4. (a) Find a formula for a bijective map \(g\) between the upper half-plane \(U = \{z \in \mathbb{C} \mid \text{Im } z > 0\}\) and the unit disk \(D = \{z \in \mathbb{C} \mid |z| < 1\}\) so that both \(g\) and \(g^{-1}\) are holomorphic.

(b) Assume that \(f\) is an analytic function from \(U\) to itself such that \(f(i) = i\). Prove that \(|f'(i)| \leq 1\).

5. We say that a sequence of real numbers \((a_n)_{n=1}^\infty\) increases subexponentially if for every \(\epsilon > 0\) there exists \(M < \infty\) such that \(|a_n| \leq Me^{\epsilon n}\) for all \(n\).

If \((f_n)_{n=1}^\infty\) is a sequence of continuous functions from \(\mathbb{R}\) to \(\mathbb{R}\), prove that the set

\[
\{ x \in \mathbb{R} \mid \text{the sequence } (f_n(x))_{n=1}^\infty \text{ increases subexponentially} \}
\]

is a Borel set.

6. (a) Find complex polynomials \(p(z)\) and \(q(z)\) so that

\[
\cos \theta = \frac{p(e^{i\theta})}{q(e^{i\theta})} \quad \text{for all } \theta \in \mathbb{R}.
\]

(b) Compute, for any \(a > 1\), the integral

\[
\int_0^\pi \frac{1}{a + \cos \theta} \, d\theta.
\]