Analysis Exam, January 2020

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that, for each $t \in \mathbb{R}$, $f^t(x) = f(t, x)$ is a Borel function from $\mathbb{R}$ to $\mathbb{R}$, and that $f^x(t) = f(t, x)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ for every $x \in \mathbb{R}$. Assume further that there is an integrable $g : \mathbb{R} \to \mathbb{R}$ with $|f(t, x)| \le g(x)$ for every $x, t \in \mathbb{R}$. Prove that the function f^t is integrable for every $t \in \mathbb{R}$ and the function $F : \mathbb{R} \to \mathbb{R}$ defined by
$$F(t) = \int_{\mathbb{R}} f^t(x)dx = \int_{\mathbb{R}} f(t, x)dx$$
is continuous.

2. Prove that
$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$$
is an entire function of z and that $f(0) = 1$ and $f'(0) = -\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. For each of the following, either prove the statement or describe a counterexample:
   (a) A Borel subset of $\mathbb{R}$ that does not contain any closed interval of the form $[a, b]$ with $a < b$ has Lebesgue measure 0.
   (b) Every function $f \in L^2([0, 1], dx)$ is in $L^1([0, 1], dx)$.
   (c) Every function $f \in L^1(\mathbb{R}, dx)$ is in $L^2(\mathbb{R}, dx)$.
   (d) If $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then the set $f([0, 1])$ has zero outer Lebesgue measure in $\mathbb{R}^2$.

4. Let $a > 0$. Evaluate the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4 + a^4} dx$.

5. Recall that a function $f : X \to \mathbb{R}^n$ with $X \subset \mathbb{R}^n$ is called Lipschitz with Lipschitz constant $L > 0$ if, for every $x, y \in X$, $|f(x) - f(y)| \le L|x - y|$. Here, $|\cdot|$ is the usual Euclidean norm.
   (a) Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of Lipschitz functions from $[0, 1]$ to $\mathbb{R}$ all with Lipschitz constant $L > 0$. Prove that there is a subsequence of $\{f_n\}$ that converges uniformly to a Lipschitz function or to $+\infty$ or to $-\infty$.
   (b) Suppose that $X \subset \mathbb{R}^n$ has outer Lebesgue measure $M \ge 0$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function with Lipschitz constant $L > 0$. Prove that $f(X)$ has outer Lebesgue measure bounded by CL^nM for some constant C which depends on n only.

6. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function.
   (a) If $z_1 \in \mathbb{D}$ and $f(z_1) = 0$, prove that
$$|f(z)| \le \frac{z - z_1}{1 - \bar{z}_1 z}\ , \qquad \forall z \in \mathbb{D}.$$
   (b) If $z_1, z_2 \in \mathbb{D}$, $z_1 \ne z_2$, and $f(z_1) = f(z_2) = 0$, prove that
$$|f(z)| \le \frac{z - z_1}{1 - \bar{z}_1 z} \times \frac{z - z_2}{1 - \bar{z}_2 z}\ , \qquad \forall z \in \mathbb{D}.$$