Analysis Exam, January 2018

1. (a) Show that if $f : \mathbf{C} \to \mathbf{C}$ is holomorphic and $\sup_{z \in \mathbf{C}} |\mathcal{I}m\, f| < \infty$, then f is a constant.

(b) Does there exist a nonconstant holomorphic g on $\mathbf{C} \setminus \{0\}$ with $\sup_{z \in \mathbf{C}\setminus\{0\}} |\mathcal{I}m\, f| < \infty$? Give an example or explain why not.

2. Suppose X is a nonempty set, and f_n is a sequence of nonnegative real-valued functions on X converging uniformly to some function $f : X \to \mathbf{R}$. Are the following 2 statements necessarily true?

(a) The sequence f_n^2 is uniformly convergent to f^2.

(b) The sequence $\sqrt{f_n}$ is uniformly convergent to $\sqrt{f}$.

Explain each answer with a proof or counterexample.

3. For $a \in \mathbf{R}$, find

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\, dx \ .$$

4. Suppose that $f : [0, \infty) \to [0, \infty)$ is a continuous function in L^2.

(a) Show that if f is decreasing, then $\lim_{x\to\infty} \sqrt{x} f(x) = 0$. Hint: Consider the integral of f^2 from a to $2a$.

(b) Show that, without the decreasing assumption, it is possible to have a continuous L^2 function f with $\limsup_{x\to\infty} \sqrt{x} f(x) = \infty$.

5. Suppose D is the open unit disk $\{z \in \mathbf{C} \ : \ |z| < 1\}$.

Find $\int_{\partial D} \frac{f'(z)}{f(z)}\, dz$ where $f(z) = z^5 - z + 2$.

6. Suppose $f \in L^1([0,1])$ and $g(c) = \int_0^1 |f(t) - c|\, dt$ for $c \in \mathbf{R}$

(a) Show that $\int_0^1 |f(t)|\, dt = \int_0^\infty \text{meas}\,\{t \ : \ |f(t)| > s\}\, ds$.

(b) Show that $\lim_{c\to\pm\infty} g(c) = \infty$.

(c) Use (a) to show that g is absolutely continuous on $\mathbf{R}$ and find a formula for $g'(c)$ for a.e. $c \in \mathbf{R}$.