RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2017

This is a 4 hour, closed book, closed notes exam. **Justify all of your work, as much as time allows.** Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let $n \geq 1$. Prove that $\mathbb{C}P^n$ has the structure of a smooth $2n$-dimensional manifold.

2. Let F be a free group of rank n and let S be a subgroup of F of finite index d.
   (a) Prove that S is a free group.
   (b) Calculate the rank (as a free group) of S in terms of n and d.

3. Consider $Y = \mathbb{R}P(2) \vee S^1$, that is to say, the wedge or one-point union of the real projective plane and a circle.
   (a) Calculate $\pi_1(Y)$
   (b) Describe all of the 2-fold covering spaces of Y.
   (c) Prove that no covering space of Y is homeomorphic to an orientable surface.

4. Let Y be the 3-manifold obtained by gluing $S^1 \times D^2$ to $S^1 \times D^2$ via $f : S^1 \times S^1 \to S^1 \times S^1$, the map on the boundary, defined by $f(x, y) = (2x + y, x + y)$. That is,
$$Y = S^1 \times D^2 \cup_f S^1 \times D^2 = (S^1 \times D^2 \sqcup S^1 \times D^2)/\sim$$
   where $(x, y) \sim f(x, y)$ and we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$.
   (a) Compute $H_p(Y; \mathbb{Z})$ and $H^p(Y; \mathbb{Z})$ for all p.
   (b) Prove there is another closed 3-manifold with the same homology groups as Y but is not homotopy equivalent to Y.

5. Let M be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_5 * \mathbb{Z}_5$ and $\chi(M) = 5$.
   (a) Compute $H_p(M; \mathbb{Z})$ for all p.
   (b) Prove that M is not homotopy equivalent to a CW complex with no 3-cells.

6. Let X and Y be closed, oriented n-dimensional manifolds and let $f : X \to Y$ be a continuous map. Suppose $\beta_p(X) < \beta_p(Y)$ for some $p > 0$.
   (a) Prove that $f^* : H^p(Y; \mathbb{Q}) \to H^p(X; \mathbb{Q})$ has a non-trivial kernel.
   (b) Show that f is a degree zero map.