RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2018

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let $M_{2n}(\mathbb{R})$ be the space of $2n \times 2n$ real matrices. Consider the following matrix in block form:
$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$
where I_n is the $n \times n$ identity matrix. Show that the subspace $S = \{A \in M_{2n}(\mathbb{R}) : A^t J_n A = J_n\}$ is a smooth submanifold of $M_{2n}(\mathbb{R})$, and compute its dimension (A^t denotes the transpose of A).

2. Let $W = S^1 \vee S^1$ be the wedge of 2 circles. Describe four distinct connected 3-fold covering spaces of W including at least one irregular cover. In each case, give the group of covering transformations, say whether or not the covering is regular and give the corresponding subgroup of $\pi_1(W)$.

3. Prove or disprove: If X is a path-connected finite CW complex and $x_0 \in X$, then $\pi_i(X, x_0)$ is finitely generated for all i.

4. Let M and N be connected, closed, orientable 3-dimensional manifolds.
   (a) Suppose that $\pi_1(M) \cong \pi_1(N)$. Prove that $H_i(M) \cong H_i(N)$ for all i.
   (b) Prove or disprove: If $H_i(M) \cong H_i(N)$ for all i then $\pi_1(M) \cong \pi_1(N)$.
   (c) Prove or disprove: There is a connected, closed, orientable 3-dimensional manifold M with $H_2(M) \cong \mathbb{Z}_5$.
   (d) Prove or disprove: If $H_i(M) \cong H_i(N)$ for all i then $H^*(M) \cong H^*(N)$ where $H^*(M)$ is the cohomology *ring* of M with $\mathbb{Z}$ coefficients.
   (e) Prove or disprove: There are connected, closed, orientable 3-dimensional manifolds M and N that are not homotopy equivalent but have $H_i(M) \cong H_i(N)$ for all i.

5. Let X be the space obtained from a solid octagon by identifying sides as shown Figure 1 below.

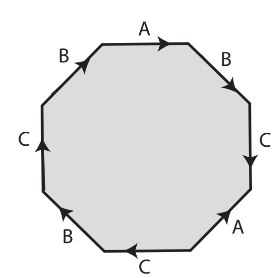

FIGURE 1

   (a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex.
   (b) Give a presentation for $\pi_1(X)$.
   (c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{Q})$ for all $n \geq 0$.

6. (a) Prove that $\mathbb{C}P^n$ is orientable for all $n \geq 0$.
   (b) For $k \geq 0$, prove that there is no orientation-reversing (that is degree -1) map $f : \mathbb{C}P^{2k} \to \mathbb{C}P^{2k}$.
   (c) For each $k \geq 0$, prove there is an orientation-reversing map $f : \mathbb{C}P^{2k+1} \to \mathbb{C}P^{2k+1}$.