Algebra Qualifying Exam

Rice University Mathematics Department

August 21, 2008

1. Let R be an integral domain with fraction field K.

   a. Assume R is a unique factorization domain. Suppose that the monic polynomial

   $$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in R[x]$$

   has a root $\alpha \in K$. Show that $\alpha \in R$.

   b. If $R = \mathbb{R}[u,v]/\langle v^2 - u^3 \rangle$ show that $p(x) = x^2 - u$ has a root over K but not over R.

2. Consider a group homomorphism $\ : \mathbb{Z}^n \to \mathbb{Z}^m$.

   a. Show that kernel( ) $= 0$ or is isomorphic to $\mathbb{Z}^k$ where $n - m \leq k \leq n$.

   b. For the specific homomorphism $\ : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by

   $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

   show that the cokernel $\mathbb{Z}^3/\psi(\mathbb{Z}^3)$ is a cyclic group of order 12.

3. Show there exists a nonabelian group G of order 21. Which Sylow subgroups of G are normal? How many elements of order three does G have?

4. Let A be an $n \times n$ complex matrix. Show there exist $n \times n$ matrices D and N such that $A = D + N$ and the following conditions are satisfied:

- D is diagonalizable, i.e., it can be diagonalized after a suitable change of basis;
- N is nilpotent, i.e., some power of N is zero;
- D and N commute.

Hint: An example of such a decomposition is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

5. Let K denote the splitting field of the polynomial $x^5 - 2$ over $\mathbb{Q}$.

   a. Show that $x^5 - 1$ splits over K.

   b. Compute the degree $[K : \mathbb{Q}]$.

   c. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is not abelian.

6. Consider the ideal

$$J = \langle x - t^2, y^2 - t^3 \rangle \subset \mathbb{Q}[x, y, t].$$

Show that the intersection

$$J \cap \mathbb{Q}[x, y]$$

is generated by $x^3 - y^4$ as an ideal over $\mathbb{Q}[x, y]$.