Instructions:

• You have 3 hours to complete this exam. Attempt all six problems.
• The use of books, notes, calculators, or other aids is not permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.

(1) Let $G$ be a finite group of odd order.
   (a) Show that any subgroup $H \subset G$ of index three is normal.
   (b) Show by example that a subgroup of index five need not be normal.

(2) Are any of the rings $\mathbb{Z}[\sqrt{5}]/(2)$, $\mathbb{Z}[\sqrt{-2}]/(2)$, and $\mathbb{Z}[\sqrt{-3}]/(1 + \sqrt{-3})$ isomorphic to one another? Justify.

(3) Suppose that $K/\mathbb{Q}$ is a Galois extension with $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Prove that $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $d_1, d_2 \in \mathbb{Q}$ have the property that none of $d_1$, $d_2$, or $d_1d_2$ is a square in $\mathbb{Q}$.

(4) Let $V$ be a finite-dimensional vector space over a field of characteristic 0. Suppose that $T: V \to V$ is a linear operator such that the trace of $T^k$ is 0 for all integers $k > 0$.
   (a) Prove that the determinant of $T$ is 0, and conclude that $T$ is not surjective (hint: apply Cayley-Hamilton).
   (b) Set $W = T(V)$, so $W$ is a proper subspace of $V$ which is preserved by $T$. Define $S: W \to W$ to be the restriction of $T$ to $W$. Prove that the trace of $S^k$ is 0 for all integers $k > 0$.
   (c) Prove that $T$ is nilpotent, i.e., that $T^p = 0$ for some integer $p > 0$.

(5) Let $R = \mathbb{Q}[x, y]$ and let $M$ and $N$ be finitely generated modules over $R$.
   (a) Show that if $M$ and $N$ are flat then $M \otimes_R N$ is flat as well.
   (b) If $M$ and $N$ are projective does it follow that $M \otimes N$ is projective as well?

(6) Let $R$ be a principal ideal domain, and let $Q$ be an $R$-module.
   (a) Show that $Q$ is injective if and only if $rQ = Q$ for every nonzero $r \in R$.
   (b) Can a nonzero finitely generated abelian group, considered as a $\mathbb{Z}$-module, be injective?