ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2014

Instructions:

• You have 4 hours to complete this exam. Attempt all six problems.
• The use of books, notes, calculators, or other aids is not permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.

(1) Classify, up to isomorphism, groups of order 21.

(2) Let $S = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$, and let $GL_4(\mathbb{Q})$ denote the group of invertible $4 \times 4$ matrices with rational entries. Write Id for the identity of the group $GL_4(\mathbb{Q})$.

(a) Show that if $A \in GL_4(\mathbb{Q})$ has order $n$, then $n$ is an element of the set $S$. [Recall that the order of a matrix $A$ is the smallest positive integer $n$ satisfying $A^n = \text{Id.}$]

(b) For each of the nine elements $n \in S$, write down a matrix $A \in GL_4(\mathbb{Q})$ of order $n$.

(3) Let $\mathbb{F}$ be a finite field, and write $\mathbb{Q}$ for the field of rational numbers.

(a) Which groups of order 6 are Galois groups of polynomials with coefficients in $\mathbb{F}$?

(b) For each group $G$ of order 6, determine a polynomial $f_G(x) \in \mathbb{Q}[x]$ with Galois group isomorphic to $G$ and find all the subfields of the splitting field of $f_G$.

(4) Let $\mathbb{F}_2$ denote a field with 2 elements, and let $\mathbb{F}_4$ be an extension of degree two of $\mathbb{F}_2$. Determine the set of prime ideals of the ring $\mathbb{F}_4 \otimes_{\mathbb{F}_2} \mathbb{F}_4$.

(5) (a) Write down polynomial equations for the image of the circle

$$C = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

under the mapping

$$(x, y) \mapsto (xy, x^2 - y^2).$$

(b) Let $f(x, y) \in \mathbb{R}[x, y]$ be a polynomial such that

$$f(x, y), x^2 + y^2 - 1) = \mathbb{R}[x, y].$$

Show that for each point $p \in C$, we have $f(p) \neq 0$.

(c) Let $f(x, y) \in \mathbb{R}[x, y]$ be a polynomial such that $f$ has no zeros on $C$. Does it follow that

$$f(x, y), x^2 + y^2 - 1) = \mathbb{R}[x, y]?$$

(6) Let $R = \mathbb{C}[x, y], I = \langle x, y \rangle$, and $Q = R/I$. For each nonnegative integer $i$, compute the groups Tor^R_i(R, Q)$ and Tor^R_i(Q, Q). State explicitly which of these groups are zero.