ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2015

Instructions:

- You have 3 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: May 12th, 2015.

(1) Show that the symmetric group S_6 contains a subgroup isomorphic to the symmetric group S_5 that acts transitively on the set $\{1, 2, 3, 4, 5, 6\}$. [Hint: consider 5-Sylow subgroups of S_5.]

(2) Let $\phi\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the map of abelian groups associated to the matrix

$$\begin{pmatrix} 1 & 4 & -3 \\ 2 & 5 & -2 \\ 1 & 16 & 9 \end{pmatrix}.$$

Is the cokernel $\mathbb{Z}^3/\phi(\mathbb{Z}^3)$ a cyclic group? What is its order?

(3) Consider the polynomial $f(x) = x^3 + 7x + 5$.

(a) Describe the splitting field of $f(x)$ over the field with two elements.

(b) Describe the splitting field of $f(x)$ over the field with thirteen elements.

(c) Compute the Galois group for the splitting field of $f(x)$ over the rational numbers.

Carefully justify your answers.

(4) Let p denote a prime number, and let $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices over the field with p elements. Prove that G has only one conjugacy class of order p elements.

(5) Let p and q be prime integers.

(a) Construct a surjective $\mathbb{Q}$-algebra homomorphism

$$\phi\colon \mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q}) \to \mathbb{Q}(\sqrt{p}, \sqrt{q}).$$

(b) Prove that if $p \neq q$ then ϕ is an isomorphism.

(c) Suppose that $p = q$. Find a basis for the kernel of ϕ as a $\mathbb{Q}$-vector space.

(6) Let $A \subseteq B$ be rings, B integral over A. Let S be a multiplicatively closed subset of A.

(a) Prove that $a \in A$ is invertible as an element of A if and only if it is invertible as an element of B.

(b) Prove that $S^{-1}B$ is integral over $S^{-1}A$.