ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2017

Instructions:

• You have four hours to complete this exam. Attempt all six problems.
• The use of books, notes, calculators, or other aids is not permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.

Date: May 9, 2017.
(1) Classify, up to isomorphism, groups of order 21.

(2) (a) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible, has degree 4, and has exactly two real roots. What are the possibilities for the Galois group of the splitting field of $f(x)$? Justify.
(b) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible, has degree 3, and has negative discriminant. What are the possibilities for the Galois group of the splitting field of $f(x)$? Justify.

(3) Let $i$ denote a primitive fourth root of unity.
   (a) Determine the irreducible polynomial for $a = i + \sqrt{2}$ over $\mathbb{Q}$.
   (b) Determine the automorphism group of $\mathbb{Q}(a)$. Is $\mathbb{Q}(a)$ a Galois extension of $\mathbb{Q}$?

(4) Let $M_2(\mathbb{C})$ denote the complex vector space of $2 \times 2$ matrices. Let
   
   \[
   A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
   \]

   and let $T: M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the linear transformation defined by
   
   $T(X) = XA - AX$.

   Determine the Jordan canonical form for $T$.

(5) Let $A$ be a commutative ring with unit, and let $S \subset A$ be a multiplicatively closed subset.
   (a) Suppose that $A$ is Noetherian. Is the localization $S^{-1}A$ Noetherian?
   (b) Suppose instead that $S^{-1}A$ is Noetherian. Must the original ring $A$ be Noetherian?

(6) Let $f: A \to B$ be an injective map between commutative rings with unit that makes $B$ an integral extension of $A$. Prove that the induced map on prime spectra $f^*: \text{Spec}(B) \to \text{Spec}(A)$ is closed, i.e., it maps closed sets to closed sets.

   [Hint: First reduce the problem to proving that $f^*(\text{Spec}(B))$ is closed in $\text{Spec}(A)$.]