Algebra Qualifying Exam, January, 2013

Time limit 3 hours. Use no books, notes, calculators, or other aids. Do as many problems as you can in any order. Concentrate on good exposition with complete and well reasoned arguments for all steps including calculations.

1. Let G be a finite group of odd order . Show that a subgroup of index 3 in G must be normal.

2. Let F be a field and L the splitting field of $p(x) = x^4 - 2$ over F.
(a) Assume $F = \mathbb{Q}$, the field of rational numbers. Compute the Galois group $G(L/\mathbb{Q})$.
(b) Now assume F is the field with 101 elements. What is $G(L/F)$?
(c) Suppose F is the field with 23 elements. Factor $p(x)$ into irreducible polynomials over F, proving each factor is irreducible.

3. Let k be a field, $S = k[s]$ the polynomial ring over k, and $R = k[s^3, s^4] \subset S$.
(a) Is S flat as an R-module?
(b) Let $I = \langle s^3, s^4 \rangle \subset R$ denote the ideal generated by the indicated polynomials. Show that I is a torsion-free R-module, but is not projective over R.
(c) Let Q be a prime ideal of S and $P = Q \cap R$. Show that P is a prime ideal as well.
(d)Determine which primes Q of S have the property that the localization S_Q is flat over the localization R_P.

4. Let R be a principal ideal domain and M be a free R-module of rank $n < \infty$.
(a) Show that every submodule of M is free of rank $\leq n$.
(b) Show that a quotient module Q of M is free if and only if it is torsion free – i.e. if $q \in Q$ and $rq = 0$ for some $r \in R - \{0\}$ then $q = 0$.

5. Consider the ideal $I = (t^2 + t - x, t - 1 - y)$ in $\mathbb{Q}[t, x, y]$.
(a). Show that $\{t - y - 1, x - y^2 - 3y - 2\}$ is a Groebner Basis for I for the lexicographic order $t > x > y$.
(b) Compute the kernel of the ring homomorphism $\mathbb{Q}[x, y] \to \mathbb{Q}[t]$ given by $x \mapsto t^2 + t$ and $y \mapsto t - 1$.

6. Let L be a finite extension of a field K and M a finite extension of L. For each of the extensions $L/K, M/L, M/K$ is it possible to choose the fields so that the extension in question is not Galois while the other two extensions are each Galois? Explain thoroughly.

PLEDGE