Analysis Preliminary Exam, Fall 2011

1. Suppose f is a Lebesgue integrable function on $(0, a)$ and $g(x) = \int_x^a f(t)\, dt$ for $0 < x < a$. Prove that g is also a Lebesgue integrable function on $(0, a)$ and

$$\int_0^a g(x)\, dx \;=\; \int_0^a x\, f(x)\, dx\ .$$

2. Determine all entire holomorphic functions f which satisfy the condition $f(z) = f(z^2)$ for all $z \in \mathbf{C}$.

3. Let f be a real integrable function on the interval $(0, 1)$. Prove that:

$$\int_0^1 f(x)\, dx \;=\; 0 \qquad \Longleftrightarrow \qquad \int_0^1 |1 \,+\, tf(x)|\, dx\ \geq 1\ \ \textit{for all}\ \ t \in \mathbf{R}\ .$$

Hint: To prove the one implication $\Longleftarrow$, proceed by calculating

$$\lim_{t\to 0} \frac{\int_0^1 |1 \,+\, tf(x)|\, dx \;-\; 1}{t}\ .$$

4. If $f(t)$ is continuous and uniformly bounded for all $t \geq 0$, show that, for $\mathcal{R}e(z) > 0$, the function(Laplace transform)

$$g(z) \;=\; \int_0^\infty f(t)\, e^{-z\, t}\, dt$$

is holomorphic for $\mathcal{R}e(z) > 0$. Hint: Show that, for each positive n, $g_n(z) \;=\; \int_0^n f(t) e^{-zt} dt$ is an entire function and that the sequenceg_n converges uniformly on each region $\{z\ :\ \mathcal{R}e(z) > c\}$ for $c > 0$.

5. Suppose $f_n(x) = \sqrt{n} e^{-nx}$ for $x \in \Omega = (0, 1)$. Prove your answers to each of the following questions:

(a) Does $f_n(x) \to 0$ as $n \to \infty$ for each $x \in \Omega$?

(b) Does f_n converge uniformly on Ω as $n \to \infty$?

(c) Does $\|f_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$?

(d) Does $\int_\Omega f_n(x) g(x)\, dx \to 0$ as $n \to \infty$ for each $g \in L^2(\Omega)$?

(e) Does $\int_\Omega f_n(x) h(x)\, dx \to 0$ as $n \to \infty$ for each $h \in L^1(\Omega)$?

6. Suppose that f be a continuous map from the closed disk $\bar{D} = \{z\ :\ |z| \leq 1\}$ into the open disk $D = \{z\ :\ |z| < 1\}$ and that f is analytic on the open disk D. Prove that f has exactly one fixed point (that is, a point a such that $f(a) = a$.)