(1) Let f be a complex-valued function defined on $\mathbb{R}$ which is Lebesgue measurable and has

$$\int_{-\infty}^{\infty} |f(x)|\, dx \ < \ \infty \ .$$

Consider the infinite series

$$g(x) \ = \ \sum_{n=-\infty}^{\infty} f(x+n) \ .$$

(a) Prove that for almost every $x \in \mathbb{R}$, the series defining $g(x)$ is absolutely convergent. (Therefore $g(x)$ is well defined almost everywhere, and periodic with period 1.)

(b) Prove that $\int_0^1 |g(x)|\, dx \ \leq \ \int_{-\infty}^{\infty} |f(x)|\, dx$.

(2) Let $z_0 \in \mathbb{C}$ with $z_0 \neq 1$. For every integer n, calculate the counterclockwise integral

$$\int_{|z|=1} (z - z_0)^n \, dz \ .$$

(3) Consider a sequence of Lebesgue measurable functions $f_n : [0,1] \to [0,\infty)$, and recall that Fatou's Lemma states that

$$\int_{[0,1]} \left(\liminf_{n\to\infty} f_n \right) dx \ \leq \ \liminf_{n\to\infty} \int_{[0,1]} f_n \, dx \ .$$

(a) Give an example of such a sequence for which

$$\int_{[0,1]} \left(\liminf_{n\to\infty} f_n \right) dx \ < \ \liminf_{n\to\infty} \int_{[0,1]} f_n \, dx \ .$$

(b) Give a (different) example of such a sequence for which

$$\int_{[0,1]} \left(\limsup_{n\to\infty} f_n \right) dx \ > \ \limsup_{n\to\infty} \int_{[0,1]} f_n \, dx \ .$$

(4) (a) State the Argument Principle, which is an integral formula for the number of zeroes (counting multiplicities) of a complex-valued function f which is continuous on $|z| \leq 1$, holomorphic on $|z| < 1$, and nonvanishing on $|z| = 1$.

(b) Let D be a connected open subset of $\mathbb{C}$ and consider a sequence f_n of holomorphic functions on D which converge uniformly to a holomorphic function f. Assuming that each f_n is injective (one-to-one), prove that

either f is also an injection or f is a constant.

Turn over for problems (5) and (6).

(5) This problem concerns the space $L^2(\mathbb{R})$ consisting of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\| \;=\; \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2} \;<\; \infty \;.$$

Let $f, g \in L^2(\mathbb{R})$ and assume $\|f\| > 0$. Prove that the limit

$$\lim_{t\to\infty} \frac{\|f + tg\| - \|f\|}{t}$$

exists, and calculate it .

(6) Again let D be a connected open subset of $\mathbb{C}$, and suppose $z_0 \in D$.
Denote by U the open unit disc : $U = \{z \in \mathbb{C} \;\big|\; |z| < 1\}$, and let

$$\mathcal{H} \;=\; \{\text{ holomorphic } f : D \to U \;\} \;.$$

Define

$$P(z_0) \;=\; \sup\{|f'(z_0)| \;\big|\; f \in \mathcal{H}\} \;.$$

(a) Prove that $0 \le P(z_0) < \infty$.

(b) Give an example of a D, z_0 for which $P(z_0) = 0$.

(c) Prove that

$$P(z_0) \;=\; \sup\{|f'(z_0)| \;\big|\; f \in \mathcal{H} \text{ and } f(z_0) = 0\} \;.$$

(d) In case $D = U$ and $z_0 = 0$, calculate $P(z_0)$.