(1) Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $a, b \in \mathbb{D}$ with $f(a) = b$. Prove that

$$f'(a) \leq \frac{1 - |b|^2}{1 - |a|^2}\,.$$

(Here $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.)

(2) Suppose that f is a Lebesgue measurable function on the interval $[0, 1]$ and $g(x) = \sqrt{x}$ for $x \in [0, 1]$. Prove:

(a) $\|f \circ g\|_{L^1} \leq 2\|f\|_{L^1}$.

(b) $\|f \circ g\|_{L^1} \leq \frac{7}{6}\|f\|_{L^2}$.

Here $\|f\|_{L^p} = (\int_0^1 |f(x)|^p \, dx\,)^{1/p}$.

(3) Evaluate the integral

$$\int_0^\infty \frac{\sin ax}{x(1 + x^2)}\ ,$$

where $a > 0$.

(4) Prove that for any (real-valued) $f \in L^1([0, 1])$, there exists a number $c \in [0, \frac{1}{2})$ such that

$$\int_c^{c+\frac{1}{2}} f(x)\, dx \ = \ \frac{1}{2} \int_0^1 f(x)\, dx\ .$$

(5) How many zeros does the function $f(z) = 9z^{10} - e^{2z}$ have inside the unit circle? Are the zeros distinct?

(6) Compute:

(a) $\lim_{n\to\infty} \int_0^\infty \frac{x^{n-2}}{1+x^n}\, dx$

(b) $\lim_{n\to\infty} n \int_0^\infty \frac{\sin y}{y(1+n^2y^2)}\, dy$. Hint: Substitute $x = ny$.