(1) Suppose \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic and \( a, b \in \mathbb{D} \) with \( f(a) = b \). Prove that
\[
f'(a) \leq \frac{1 - |b|^2}{1 - |a|^2}.
\]
(Here \( \mathbb{D} \) denotes the open unit disc in \( \mathbb{C} \).)

(2) Suppose that \( f \) is a Lebesgue measurable function on the interval \([0, 1]\) and \( g(x) = \sqrt{x} \) for \( x \in [0, 1] \). Prove:
(a) \( \|f \circ g\|_{L^1} \leq 2\|f\|_{L^1} \).
(b) \( \|f \circ g\|_{L^1} \leq \frac{7}{6}\|f\|_{L^2} \).
Here \( \|f\|_{L^p} = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \).

(3) Evaluate the integral
\[
\int_0^\infty \frac{\sin ax}{x(1 + x^2)} \, dx,
\]
where \( a > 0 \).

(4) Prove that for any (real-valued) \( f \in L^1([0, 1]) \), there exists a number \( c \in [0, \frac{1}{2}] \) such that
\[
\int_c^{c+\frac{1}{2}} f(x) \, dx = \frac{1}{2} \int_0^1 f(x) \, dx.
\]

(5) How many zeros does the function \( f(z) = 9z^{10} - e^{2z} \) have inside the unit circle? Are the zeros distinct?

(6) Compute:
(a) \( \lim_{n \to \infty} \int_0^\infty \frac{a^n - 2}{1 + x^n} \, dx \)
(b) \( \lim_{n \to \infty} n \int_0^\infty \frac{\sin y}{y(1 + n^2 y^2)} \, dy \). Hint: Substitute \( x = ny \).