1. Given that \( \int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi} \), calculate
\[ \int_{-\infty}^{\infty} e^{-t^2 + tw} \, dt \]
for any complex number \( w \).

2. Let \( f \) be a Möbius (also called linear fractional) transformation
\[ f(z) = \frac{az + b}{cz + d} \]
where \( a, b, c, d \) are complex numbers with \( ad - bc \neq 0 \).
It is known that in general if \( K \) is a circle or straight line in the complex plane \( \mathbb{C} \), then \( f(K) \) is also a circle or a straight line.

Now suppose that there exists a circle \( K \) with center \( z_0 \) such that \( f(K) \) is a circle with center \( f(z_0) \).

What can you conclude about \( f \) ?
3. Let \( D = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) and suppose that \( f \) is a holomorphic function defined on \( D \). Suppose also that in the sense of integration on \( \mathbb{R}^2 \)

\[
\iint_D |f(x+iy)| \, dx \, dy < \infty.
\]

What can you conclude about the nature of the singularity of \( f \) at \( z = 0 \)?

(Is it removable? If so, explain. If not, what is the nature of this singularity?)

4. Prove the "Riemann-Lebesgue lemma" in detail; namely, if \( f \) is Lebesgue integrable on \( \mathbb{R} \), then

\[
\lim_{x \to \infty, x \in \mathbb{R}} \int_{-\infty}^{\infty} f(t) e^{itx} \, dt = 0.
\]

5. The Weierstrass approximation theorem states that every continuous real-valued function on a compact interval \([a, b] \subset \mathbb{R}\) can be uniformly approximated by polynomials.

Given that, suppose \( f \) is a \( C^1 \) real-valued function on \([a, b]\). That is, \( f \) is differentiable on \([a, b]\) and its derivative \( f' \) is continuous on \([a, b]\). Let \( \varepsilon > 0 \). Then prove that there exists a polynomial \( P = P(x) \) such that

\[
\sup_{a \leq x \leq b} |f(x) - P(x)| + \sup_{a \leq x \leq b} |f'(x) - P'(x)| < \varepsilon.
\]