1. Let \( f \geq 0 \) be a measurable function defined on \([0, \infty)\) such that
\[
\int_0^x f(t) \, dt \leq e^x \quad \text{for all} \quad 0 \leq x < \infty.
\]
Suppose that \( a > 1 \). Prove that
\[
\int_0^\infty f(t) e^{-at} \, dt < \infty.
\]

2. Suppose that \( f \) is a holomorphic function defined on some neighborhood of the origin. Suppose that \( f \) satisfies the equation
\[
f(2z) = 2f(z) f'(z) = 2f(z) \frac{df}{dz}
\]
for all \( |z| < \varepsilon \) for some \( \varepsilon > 0 \). Prove that there exists an entire holomorphic function which is equal to \( f \) on some neighborhood of the origin.

3. Let \( f \) and \( g \) be real valued functions belonging to \( L^2(\mathbb{R}) \) (they are measurable and their squares are integrable on \( \mathbb{R} \)). Let \( h \) be the convolution of \( f \) and \( g \). That is, \( h \) is the function defined by
\[
h(x) = \int_{-\infty}^{\infty} f(x-t) g(t) \, dt.
\]
Prove that \( h \) is a bounded continuous function on \( \mathbb{R} \), and that \( \lim_{|x| \to \infty} h(x) = 0 \).
4. Let \( f \) be the function defined by

\[
f(z) = \int_{-1}^{1} \frac{dt}{b - z},
\]

where \( z \in \mathbb{C} \) and \( z \) is not a real number in the interval \([-1, 1]\).

a. Prove that \( f \) is holomorphic.

b. Compute \( \lim_{y \to 0+} f(iy) \) and \( \lim_{y \to 0-} f(iy) \).

5. Construct a real valued \( C^\infty \) function on \( \mathbb{R} \) which equals 0 on \((-\infty, 0] \) and equals 1 on \([1, \infty) \).

6. Suppose \( f \) is a holomorphic function defined on the unit disk \( |z| < 1 \), and suppose \( f \) is not identically zero. Is it possible that for every \( z_0 \) such that \( |z_0| = 1 \) there exists a sequence \( \{z_1, z_2, z_3, \ldots \} \) such that \( |z_n| < 1 \) for all \( n \geq 1 \) and \( f(z_n) = 0 \) for all \( n \geq 1 \)?

Explain.