Analysis Exam, May 2012

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and both f and f' are integrable on $\mathbf{R}$. Show that $\int_{-\infty}^{\infty} f'(t)\, dt \;=\; 0$.

2. (a) Give an example of a pointwise convergent sequence of real-valued differentiable functions $g_n : [-1, 1] \to [-1, 1]$ whose derivatives $g_n'(0)$ do *not converge.*
(b) Suppose that D is the open unit disk $\{z \in \mathbf{C} \,:\, |z| < 1\}$. Prove that if $f_n : D \to D$ is a pointwise convergent sequence of *holomorphic* functions, then the derivatives f_n' converge at every point of D.

3. Suppose that $A_1, A_2, \ldots$ and $E_1, E_2, \ldots$ are Lebesgue measurable subsets of $[0, 1]$ such that $[0, 1] \subset \cup_{i=1}^{\infty} A_i$, and, for each $i = 1, 2, \ldots$, the Lebesgue measure $\lambda(A_i \cap E_j) \to 0$ as $j \to \infty$. Prove that $\lambda(E_j) \to 0$ as $j \to \infty$.

4. Suppose that f is a holomorphic function on $\mathbf{C} \setminus \{1\}$ and
$\lim_{z\to\infty} |z||f(z)| = +\infty$ and $\lim_{z\to 1} |z - 1|^2|f(z)| = +\infty$.

(a) Find one specific such f.
(b) Show that any such f is a rational function (that is, a quotient of 2 polynomials).
(c) Are all such f contained in a finite dimensional space of functions?

5. Suppose that $f(s)$ is a positive integrable function on $\mathbf{R}$.

(a) Show that the Lebesgue measure $g(s) \;=\; \lambda\{t \,:\, f(t) > s\,\}$ is a measurable function of s, and
(b) $\int_0^{\infty} g(s)\, ds \;=\; \int_{-\infty}^{\infty} f(t)\, dt$.

6. Suppose $f_k : \{z \in \mathbf{C} \,:\, |z| < 1\} \to \mathbf{C}$ is a sequence of injective holomorphic functions that converges uniformly on compact subsets to a function g.

(a) Prove that g is holomorphic.
(b) Prove that g is either injective or a constant function.