Analysis Exam, May 2013

1. Suppose f is holomorphic on a domain $\Omega \subset \mathbf{C}$ and u and v are the real and imaginary parts of f, that is, $f(x + \mathbf{i}y) = u(x, y) + \mathbf{i}v(x, y)$. Let ∇u and ∇v denote the gradients of u and v.

(a) Prove that $|\nabla u| = |\nabla v| = |\, f'\, |$.

(b) Prove that if $|\nabla u|$ is constant on Ω, then $f(z) = az + b$ for some $a, b \in \mathbf{C}$.

2. Suppose f is a differentiable function on the positive real numbers and $\alpha \in \mathbf{R}$. For the following two statements, give a proof if true or give a counter-example if false.

(a) $\limsup_{t\to\infty} t^{-\alpha}|f(t)| < \infty$ implies that $\limsup_{t\to\infty} t^{1-\alpha}|f'(t)| < \infty$.

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3. Suppose f is a rational function, all the zeros and poles of which are of even order. Prove there exists a rational function g such that $g^2 = f$.

4. Suppose f_n and f are L^1 integrable functions on $[0, 1]$. We say f_n *converges in measure to* f if, for every positive ε the Lebesgue measure of $\{t \in [0, 1] \;:\; |f_n(t) - f(t)| > \varepsilon\}$ approaches zero as $n \to \infty$.

Prove or find a counterexample to each of the following:

(a) If, as $n \to \infty$, f_n converges to f in L^1, then f_n converges in measure to f.

(b) If, as $n \to \infty$, f_n converges in measure to f, then f_n converges to f in L^1.

5. Suppose f is a holomorphic map taking $\overline{D} = \{z \in \mathbf{C} \,:\, |z| \le 1\}$ into $D = \{z \in \mathbf{C} \,:\, |z| < 1\}$. Prove that f has exactly one fixed point in D.

6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a smooth function.

(a) Prove that

$$\lim_{n\to\infty} \int_{-\pi/2n}^{\pi/2n} f(x)\frac{n}{2}\cos(nx)\,dx \;=\; f(0)\ .$$

(b) Find a function $K_n(x)$ so that

$$\lim_{n\to\infty} \int_{-\pi/2n}^{\pi/2n} f(x)\,K_n(x)\,dx \;=\; f'(0)\ .$$

7. Determine all functions f which are holomorphic in the disk $D = \{z \in \mathbf{C} \;: |z - 1| < 1\}$ and satisfy

$$f(\frac{n}{n+1}) \;=\; 1 \;-\; \frac{1}{2n^2 + 2n + 1} \quad \text{for} \quad n = 1, 2, \ldots\ .$$