1. Suppose $f$ is holomorphic on a domain $\Omega \subset \mathbb{C}$ and $u$ and $v$ are the real and imaginary parts of $f$, that is, $f(x + iy) = u(x, y) + iv(x, y)$. Let $\nabla u$ and $\nabla v$ denote the gradients of $u$ and $v$.

(a) Prove that $|\nabla u| = |\nabla v| = |f'|$.

(b) Prove that if $|\nabla u|$ is constant on $\Omega$, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

2. Suppose $f$ is a differentiable function on the positive real numbers and $\alpha \in \mathbb{R}$. For the following two statements, give a proof if true or give a counter-example if false.

(a) $\limsup_{t \to \infty} t^{-\alpha}|f(t)| < \infty$ implies that $\limsup_{t \to \infty} t^{1-\alpha}|f'(t)| < \infty$.

(b) $\limsup_{t \to \infty} t^{1-\alpha}|f'(t)| < \infty$ implies that $\limsup_{t \to \infty} t^{-\alpha}|f(t)| < \infty$.

3. Suppose $f$ is a rational function, all the zeros and poles of which are of even order. Prove there exists a rational function $g$ such that $g^2 = f$.

4. Suppose $f_n$ and $f$ are $L^1$ integrable functions on $[0, 1]$. We say $f_n$ converges in measure to $f$ if, for every positive $\varepsilon$ the Lebesgue measure of $\{t \in [0, 1] : |f_n(t) - f(t)| > \varepsilon\}$ approaches zero as $n \to \infty$.

Prove or find a counterexample to each of the following:

(a) If, as $n \to \infty$, $f_n$ converges to $f$ in $L^1$, then $f_n$ converges in measure to $f$.

(b) If, as $n \to \infty$, $f_n$ converges in measure to $f$, then $f_n$ converges to $f$ in $L^1$.

5. Suppose $f$ is a holomorphic map taking $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ into $D = \{z \in \mathbb{C} : |z| < 1\}$. Prove that $f$ has exactly one fixed point in $D$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

(a) Prove that
$$\lim_{n \to \infty} \int_{-\pi/2n}^{\pi/2n} f(x) \frac{n}{2} \cos(nx) \, dx = f(0) .$$

(b) Find a function $K_n(x)$ so that
$$\lim_{n \to \infty} \int_{-\pi/2n}^{\pi/2n} f(x) K_n(x) \, dx = f'(0) .$$

7. Determine all functions $f$ which are holomorphic in the disk $D = \{z \in \mathbb{C} : |z - 1| < 1\}$ and satisfy
$$f\left(\frac{n}{n+1}\right) = 1 - \frac{1}{2n^2 + 2n + 1} \text{ for } n = 1, 2, \ldots .$$