(1) (10 points) Compute

$$\int_0^\infty \frac{\cos 2x}{4+x^2}\,dx\ .$$

(2) (5 points) Construct a subset K of $[0;1]$ such that K is closed, K has positive Lebesgue measure, and K has empty interior.

(3) Denote $\mathbb{D} = \{z \in \mathbb{C}\ :\ |z| < 1\}$ and $Y\ =\ \{z = x + \mathbf{i}y \in \mathbb{C} : x \geq e^y\}$.
(a) (5 points) Does there exist a non-constant analytic function $f : \mathbb{D} \to Y$ such that $f(0) = 1$?
(b) (5 points) Does there exist a non-constant analytic function $f : \mathbb{D} \to Y$ such that $f(0) = 2$?
(Justify your answers.)

(4) Consider the spaces $L^p = L^p([0;1], dm)$ where $1 < p < \infty$ and dm denotes Lebesgue measure, with the associated norm $\|\cdot\|_p$.
(a) (5 points) For $1 < q < p < \infty$ and $f \in L^p$, show that $f \in L^q$ and that $\|f\|_{L^q} \leq \|f\|_{L^p}$.

(b) (5 points) For $f \in L^\infty$, show that

$$\|f\|_\infty\ =\ \lim_{p\to\infty} \|f\|_p\ .$$

(c) (5 points) Prove or disprove: $L^\infty\ =\ \bigcap_{1\leq p<\infty} L^p$.

(5) Suppose $f \in L^2 = L^2([0;1]; dm)$ as in problem (4).
(a) (5 points) Show that

$$F(x)\ =\ \int_0^x f(t)\,dm(t)$$

is a continuous function on $[0, 1]$.
(b) (5 points) Suppose that $f_n, f \in L^2$ and

$$\lim_{n\to\infty} \int_0^1 f_n(t)\,g(t)\,dm(t)\ =\ \int_0^1 f(t)\,g(t)\,dm(t)$$

for all $g \in L^2$. Show that $F_n \to F$ uniformly where $F_n(x)\ =\ \int_0^x f_n(t)\,dm(t)$ for $x \in [0;1]$.

(6) (10 points) Show that, for any $0 < r < 1$, there exists an integer $n \geq 2$ such that the polynomial

$$P_n(z)\ =\ 1 + 2z + 3z^2 + \cdots + nz^{n-1}$$

has no roots in the disk $\{z \in \mathbb{C}\ :\ |z| < r\}$.