Analysis Exam, January 2006

1. (a) Does there exist a nonconstant holomorphic function $f : \mathbf{C} \to \mathbf{C}$ satisfying $\lim_{|z|\to\infty} |z|^{-1}|f(z)| = 0$? Explain.

b) Does there exist a nonconstant holomorphic function $g : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ satisfying $\lim_{|z|\to\infty} |z|^{-1}|g(z)| = 0$ and $\lim_{|z|\to 0} |z|^{1/2}|g(z)| = 0$? Explain.

c) Does there exist a nonconstant harmonic function $h : \mathbf{C} \setminus \{0\} \to \mathbf{R}$ satisfying $\lim_{|z|\to\infty} |z|^{-1}|h(z)| = 0$ and $\lim_{|z|\to 0} |z|^{1/2}|h(z)| = 0$? Explain.

2. (a) Using the notion of a Lebesgue measurable set, give the definition of a *Lebesgue measurable function* $f : \mathbf{R} \to \mathbf{R}$.

(b) Prove that the supremum $g(x) = \sup_k f_k(x)$ of a sequence of Lebesgue measurable functions $f_k : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable.

(c) Assuming that $h(x) = \lim_{k\to\infty} f_k(x)$ and that $\int_{\mathbf{R}} |f_k(x)|\, dx = 1$ for all k, what can you say about $\int_{\mathbf{R}} |h(x)|\, dx$?

3. Suppose $f : \mathbf{C} \to \mathbf{C}$ is holomorphic.

(a) For a smooth simple closed curve $\gamma : [0, 2\pi] \to \mathbf{C}$ and a point a in the bounded component of $\mathbf{C} \setminus \gamma([0, 2\pi])$, write a formula for $f(a)$ in terms of the values $f\big(\gamma(\theta)\big)$ for $0 \le \theta \le 2\pi$.

(b) Prove that

$$|f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\mathbf{i}\theta})|\, d\theta\ .$$

(c) If $g : \mathbf{C} \to \mathbf{C}$ is a nonconstant holomorphic function and $h \equiv |f| + |g|$, show that

$$M(r) \;=\; \sup_{\{z\,:\,|z|=r\}} h(z)$$

is a strictly increasing function of $r \in [0, \infty)$.

4. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function.

(a) For each $p \in [1, \infty)$, define the L^p norm $\|f\|_{L^p}$.

(b) Show that if $\int_{\mathbf{R}} e^{|f(x)|}\, dx \;=\; 1$, then

$$\sup_{1\le p<\infty} p^{-1}\|f\|_{L^p} \;<\; \infty\ .$$

5. Suppose $f_k : \{z \in \mathbf{C} \,:\, |z| < 1\} \to \mathbf{C}$ is a sequence of injective holomorphic functions that converges uniformly on compact subsets to a function g.

(a) Prove that g is holomorphic.

(b) Prove that g is either injective or a constant function.

6. Suppose that $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $|\Phi(a) - \Phi(b)| \le |a - b|$ for all $a, b \in \mathbf{R}^2$.

(a) Show that there is a finite constant C so that , for all $E \subset \mathbf{R}^2$, the image $\Phi(E)$ has Lebesgue outer measure $\lambda^*\big(\Phi(E)\big) \le C\lambda^*(E)$.

(b) Prove that $\Phi(A)$ is Lebesgue measurable whenever A is Lebesgue measurable.