Analysis Exam, January 2007

1. (a) Give an example of a pointwise convergent sequence of smooth real-valued functions $g_n : [-1, 1] \to [-1, 1]$ whose derivatives g_n' do *not converge* at almost every point of $[-1, 1]$.
(b) Suppose $D = \{z \in \mathbf{C} : |z| < 1\}$. Prove that if $f_n : D \to D$ is a pointwise convergent sequence of holomorphic functions, then the derivatives f_n' converge at every point of D.

2. Suppose $1 \le p < q < \infty$
(a) Either prove $L^q([0, 1]) \subset L^p([0, 1])$ or find a specific function $g \in L^q([0, 1]) \setminus L^p([0, 1])$.
(b) Either prove $L^q(\mathbf{R}) \subset L^p(\mathbf{R})$ or find a specific function $g \in L^q(\mathbf{R}) \setminus L^p(\mathbf{R})$.

3. (a) Given distinct complex numbers a, b, c, d, find a holomorphic $f : \mathbf{C} \to \mathbf{C}$ such that $f(a) = b$ and $f(c) = d$.
(b) Assuming moreover that a, b, c, d lie in the unit disk D find, if possible, a holomorphic $f : D \to D$ such that $f(a) = b$ and $f(c) = d$.
(c) Find, if possible, a nonconstant holomorphic $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ with $f(1/j) = 0$ for $j = 1, 2, \cdots$.
(d) Find, if possible, a nonconstant holomorphic $f : D \to D$ with $f(1/j) = 0$ for $j = 1, 2, \cdots$.

4. Suppose that $g : [0, 1] \to [0, 10]$ is an increasing function.
(a) Show that $g_-(a) = \lim_{t \uparrow a} g(t)$ and $g_+(a) = \lim_{t \downarrow a} g(t)$ exist for all $a \in (0, 1)$ and that the set of discontinuities $E = \{a : g_-(a) \ne g_+(a)\}$ is at most countable.
(b) Try to find a good upper bound for $\sum_{a \in E} g_+^2(a) - g_-^2(a)$.

5. Suppose $f : \mathbf{C} \to \mathbf{C}$ is a holomorphic function with zeros $a_1, a_2, \cdots, a_k$ in the unit disk of multiplicities respectively $m_1, m_2, \cdots, m_k$.
(a) Find the poles, with their orders, and the residues of the meromorphic function f'/f.
(b) Describe the quantity $\sum_{j=1}^k m_j \, a_j^3$ as an integral over the unit circle of some expression involving f and its derivatives.

6. Let λ_n denote n dimensional Lebesgue measure.
(a) Suppose that $A \subset [0, 1] \times [0, 1]$ is a Lebesgue measurable with $\lambda_2(A) \ge 1/3$. Show that

$$B \;=\; \{x \in [0, 1] \;:\; \lambda_1\{y : (x, y) \in A\} \ge 1/4\} \text{ has } \lambda_1(B) \ge 1/9 \;.$$

(b) Suppose that $\alpha : [0, 1] \to \mathbf{R}$ and $\beta : [0, 1] \to \mathbf{R}$ are continuous. Together these define the curve $\gamma(t) = \big(\alpha(t), \beta(t)\big)$ in $\mathbf{R}^2$. Recall that the length of γ is given by

$$L = \sup\{\sum_{i=1}^{j} |\gamma(t_i) - \gamma(t_{i-1})| \;:\; 0 = t_0 < t_1 < \cdots < t_{j-1} < t_j = 1\} \;.$$

Prove that

$$L \;\le\; \int \#\big(\{t : \alpha(t) = x\}\big)\, dx \;+\; \int \#\big(\{t : \beta(t) = y\}\big)\, dy \;\le\; 2L \;,$$

(where $\#(E)$ is number of points, possibly infinite, in E).