1. Let $a, b, c$ be nonzero real numbers. Calculate

$$\int_{-\infty}^{\infty} \frac{dx}{(x-ia)(x-ib)(x-ic)}$$

(as usual, $i=\sqrt{-1}$).

2. a. Let $0 < a < 1$, and calculate

$$\int_{0}^{\infty} \frac{x^{a-1}}{x+1} \, dx.$$

b. Use part a to calculate

$$\int_{0}^{\infty} \frac{x^{a-1} \log x}{x+1} \, dx.$$

c. When $a=\frac{1}{2}$ part b gives the result 0. Verify directly by a change of variable that

$$\int_{0}^{\infty} \frac{x^{-\frac{1}{2}} \log x}{x+1} \, dx = 0.$$

3. Let $E \subseteq [0,1]$ and consider the indicator function

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

Prove that $\chi_E$ is Riemann integrable and

$$\int_{0}^{1} \chi_E(x) \, dx = 0$$

if and only if ____________.

(You should insert the missing condition.)
4. Let $f$ be a bounded real valued function on $[a, b]$. Prove that $f$ is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exist Lebesgue measurable simple functions $g$ and $h$ such that $g \leq f \leq h$ and $\int_a^b (h-g) \, dx < \varepsilon$.

5. Let $f_1, f_2, f_3, f_4, \ldots$ be a sequence of Lebesgue measurable functions defined on $[0, 1]$ such that

$$|f_n(x)| \leq 1 \quad \text{for all } n \geq 1 \text{ and all } 0 \leq x \leq 1,$$

and

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{exists for each } 0 \leq x \leq 1.$$

Prove that

$$\lim_{n \to \infty} \int_0^1 \frac{f_n(x)}{\sqrt{|x-h|}} \, dx = \int_0^1 \frac{f(x)}{\sqrt{x}} \, dx.$$

6. Let $f$ be the Riemann map from the unit disk $|z|^2 = x^2 + y^2 < 1$ onto the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, normalized by the two conditions $f(0) = 0$, $f'(0) > 0$.

Then of course

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{for } |z| < 1,$$

and $a_1 > 0$.

Prove that

$$\begin{cases} a_n = 0 & \text{for all even } n, \\ a_n \in \mathbb{R} & \text{for all odd } n. \end{cases}$$