Solutions to Qualifying Examination Analysis

4pm-8pm

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Problem 1

Let f be an entire function. It is given that

$$f\left(\frac{1}{k}\right) = \frac{k-1}{k^2}, k \geq 1. \tag{1}$$

Identify f and state carefully any theorem that allows you to do so.

Solution

Define $g(z) = z - z^2$. Then

$$f\left(\frac{1}{k}\right) = \frac{k-1}{k^2} = g\left(\frac{1}{k}\right), k \geq 1.$$

The entire functions f and g agree on the set $E = \left\{\frac{1}{k}, k \geq 1\right\}$. The identity theorem says that if two holomorphic functions on a region Ω agree on a set that has an accumulation point in Ω, then they agree on all of Ω. Therefore,

$$f(z) = z - z^2, z \in \mathbb{C}.$$

Problem 2

Establish the identity

$$\int_0^\infty \frac{t^{s-1}}{1+t}dt = \frac{\pi}{\sin \pi s}, 0 < s < 1.$$

Solution

Let $\log z$ be the principal branch of the logarithm on $\mathbb{C}\backslash[0,\infty)$. The boundary values of this function satisfy

$$\log t = \begin{cases} \ln t, \text{ from above} \\ \ln t + 2\pi i \text{ from below.} \end{cases}$$

Fix $\delta \in (0,1)\,, T > 1$, and consider the integral of

$$h(z) = \tfrac{e^{(s-1)\log z}}{1+z}$$

$$h(t) = \begin{cases} \tfrac{t^{s-1}}{1+t}, \text{ from above} \\ e^{2\pi i(s-1)}\tfrac{t^{s-1}}{1+t} = e^{2\pi i s}\tfrac{t^{s-1}}{1+t} \text{ from below.} \end{cases}, t > 0$$

over the closed curve

$$\alpha_{\delta,T} = C_T^+ + \gamma_{T,\delta}^- + C_\delta^- + \gamma_{\delta,T}^+$$

comprised of the circle $C_T^+ = \{|z| = T\}$, counterclockwise, the path $\gamma_{T,\delta}^-$ from T to δ on the "lower side" of the axis, the circle $C_\delta^- = \{|z| = \delta\}$, clockwise and the path $\gamma_{\delta,T}^+$ from δ to T on the "upper side" of the axis. $\alpha_{\delta,T}$ bounds a simply connected region, and therefore the Cauchy Integral Formula implies

$$\frac{1}{2\pi i}\int_{\alpha_{\delta,T}} h(z)\,dz = e^{(s-1)\log(-1)} = -e^{\pi i s}.$$

As

$$s - 1 \in (-1, 0)$$

and

$$\left|\tfrac{e^{(s-1)\log z}}{1+z}\right| = \tfrac{|z|^{-(1-s)}}{|1+z|}$$

the integral over C_T^+ (resp. C_δ^-) is bounded by $O\left(T^{-(1-s)}\right) = o(1)$ (resp. $O\left(\delta \cdot \delta^{-(1-s)}\right) = O(\delta^s) = o(1)$. Conclude that

$$-e^{\pi i s} = \tfrac{1}{2\pi i} \int\limits_{\alpha_{\delta,T}} h(z)\, dz$$

$$= \left(1 - e^{2\pi i s}\right) \tfrac{1}{2\pi i} \int\limits_{\delta}^{T} \tfrac{t^{s-1}}{1+t} dt + o(1)$$

$$= e^{\pi i s} \left(-2i \sin \pi s\right) \tfrac{1}{2\pi i} \int\limits_{\delta}^{T} \tfrac{t^{s-1}}{1+t} dt + o(1).$$

Let $T \to \infty, \delta \to 0$ and find that

$$\int\limits_0^\infty \frac{t^{s-1}}{1+t} dt = \frac{\pi}{\sin \pi s}$$

as required.

Problem 3

This problem has five parts, with one page per part.

3.1. Compute the limit

$$\lim_{z \to 0} \left(\left(\frac{\pi}{\sin \pi z} \right)^2 - \frac{1}{z^2} \right). \tag{1}$$

Remark that your answer implies that for each integer n the function

$$\left(\frac{\pi}{\sin \pi z} \right)^2 - \frac{1}{(z-n)^2}$$

has the same limit as (1) at $z = n$.

Solution

The second claim follows from the first and the periodicity of $\left(\frac{\pi}{\sin \pi z}\right)^2$. As for the first claim, Taylor's formula implies $\sin \pi z = \pi z - \frac{(\pi z)^3}{3!} + o(z^3)$, and therefore

$$\left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2} = \frac{1}{z^2}\frac{1}{\left(1-\frac{(\pi z)^2}{3!}+o(z^2)\right)^2} - \frac{1}{z^2}$$

$$= \frac{1}{z^2}\left(1 + \frac{(\pi z)^2}{3!} + o(z^2)\right)^2 - \frac{1}{z^2} = \frac{\pi^2}{3} + o(1).$$

The answer is $\frac{\pi^2}{3}$.

3.2. (Problem 3, continued.) You may assume that the series

$$\sum_{-\infty}^{\infty}\left(\frac{1}{z-n}\right)^2, z \in \mathbb{C}\backslash\mathbb{Z}$$

converges locally uniformly to a holomorphic function on the region $\mathbb{C}\backslash\mathbb{Z}$. Define a function $\varphi(z)$ on $\mathbb{C}\backslash\mathbb{Z}$ by

$$\varphi(z) = \left(\frac{\pi}{\sin \pi z}\right)^2 - \sum_{-\infty}^{\infty}\left(\frac{1}{z-n}\right)^2, z \in \mathbb{C}\backslash\mathbb{Z}. \tag{2}$$

Why does Part 3.1 allow you to conclude that φ extends to an entire holomorphic function?

Solution

The first part of the problem implies that $\varphi(z)$ is bounded in a neighborhood of each integer. By the removable singularities theorem, φ extends to be entire.

3.3. (Problem 3, continued.) Establish the inequality

$$|\sin \pi z| \geq \max\left(|\sin \pi x| \cosh \pi y, |\cos \pi x| \, |\sinh \pi y|\right), z = x+iy \in \mathbb{C}. \tag{3}$$

Use (3) to prove that $\left(\frac{\pi}{\sin \pi z}\right)^2$ is uniformly bounded on both

(A) The union of the vertical lines, $\operatorname{Re} z = n + \frac{1}{2}, n \in \mathbb{Z}$.

$$\text{and} \tag{4}$$

(B) The union of the horizontal lines, $\operatorname{Im} z = y, |y| \geq 1$.

Solution

One has

$$\sin \pi z = \sin \pi x \cos \pi i y + \cos \pi x \sin \pi i y$$

$$= \sin \pi x \cosh \pi y - i \cos \pi x \sinh \pi y.$$

Since $|\sin \pi z| \geq \max\left(|\operatorname{Re} \sin \pi z|, |\operatorname{Im} \sin \pi z|\right)$, (3) follows. (4) is then an easy consequence.

3.4. (Problem 3, continued.) You may assume that the function $\sum_{-\infty}^{\infty} \left(\frac{1}{z-n}\right)^2$ is also uniformly bounded on the lines (4). What can you then say about the function φ in (2)? (You may assume the statement in Part 3.3, even if you have been unable to prove it.)

Solution

For each $n > 0$ the lines $\operatorname{Re} z = \pm\left(n + \frac{1}{2}\right)$ and $\operatorname{Im} z = \pm n$ determine a rectangle, R_n. The maximum principle implies that $|\varphi|\,|_{R_n}$ attains its maximum on ∂R_n. As φ is uniformly bounded on $\bigcup_{n \geq 1} \partial R_n$, φ is entire and bounded. Liouville's Theorem implies that φ is constant.

3.5. (Problem 3, continued.) Prove that $\lim_{y \to \infty} \varphi(iy) = 0$. What does this imply and why?

Solution

Let $y \geq 1$ tend to $+\infty$.

$$|\varphi(iy)| = \left|\left(\frac{\pi}{\sin \pi iy}\right)^2 - \sum_{-\infty}^{\infty}\left(\frac{1}{iy-n}\right)^2\right|$$

$$= \left|-\left(\frac{\pi}{\sinh y}\right)^2 - \sum_{-\infty}^{\infty}\left(\frac{1}{iy-n}\right)^2\right| = o(1) + \left|\sum_{-\infty}^{\infty}\left(\frac{1}{iy-n}\right)^2\right|$$

$$\leq o(1) + \sum_{-\infty}^{\infty}\frac{1}{y^2+n^2} \leq o(1) + \frac{1}{y^2} + \int_{-\infty}^{\infty}\frac{dt}{t^2+y^2} = o(1).$$

Since Part 4 implies φ is constant, the constant must be zero. Therefore,

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \sum_{-\infty}^{\infty}\left(\frac{1}{z-n}\right)^2, z \in \mathbb{C}\backslash\mathbb{Z}.$$

Problem 4

NOTATIONS and ASSUMPTIONS. Denote Lebesgue measure on the unit interval by $m(\cdot)$ and in integrals by $m(dx)$. Assume that $f \geq 0$ is Lebesgue measurable on $[0,1]$. Assume in addition that

$$m(\{x|f(x) \geq t\}) = e^{-t}, t \geq 0.$$

This Problem has three parts, one page per part.

4.1. Prove that $f \in L^p(m), 0 < p < \infty$.

Solution

Define $E_n = f^{-1}([n.n+1)), n \geq 0$. By assumption,

$$m(E_n) = e^{-(n+1)} - e^{-n} = e^{-n}(e-1).$$

Then

$$f(\cdot) \leq \sum_{k=1}^{\infty} k\chi_{E_{k-1}}.$$

The ratio test implies

$$\int_{[0,1]} f^p(x)\, m(dx) \le (e-1)\sum_{k=1}^{\infty} k^p e^{-k} < \infty.$$

4.2. (Problem 4, continued.) Prove that the integral of f^p over $[0,1]$ equals the improper Riemann integral of $pt^{p-1}e^{-t}$ over $[0,\infty)$. That is, prove

$$\int_{[0,1]} f^p(x)\, m(dx) = p\int_0^{\infty} t^{p-1}e^{-t}dt, p > 0. \tag{1}$$

(Two solutions are described below.)

Solution #1

This solution relies on the Fubini Theorem. One has

$$\int_{[0,1]} f^p(x)\, m(dx) = \int_{[0,1]} \left(\int_0^{f(x)} pt^{p-1}dt \right) m(dx)$$

$$= \int_{[0,1]} \left(\int_0^{\infty} pt^{p-1}\chi_{[0,f(x)]}(t)\, dt \right) m(dx) = \int_{[0,1]} \left(\int_0^{\infty} pt^{p-1}\chi_{[t,\infty)}(f(x))\, dt \right) m(dx)$$

$$\overset{\text{Fubini}}{=} \int_0^{\infty} pt^{p-1} \left(\int_{[0,1]} \chi_{[t,\infty)}(f(x))\, m(dx) \right) dt = \int_0^{\infty} pt^{p-1} m(\{x | f(x) \ge t\})\, dt$$

$$\overset{\text{Fubini}}{=} \int_0^{\infty} pt^{p-1}e^{-t}dt.$$

The last integral has continuous integrand (on $(0,\infty)$ when $p \in (0,1)$). The monotone convergence theorem implies

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{n} pt^{p-1}e^{-t}dt = \int_0^{\infty} pt^{p-1}e^{-t}dt.$$

The integrals on the left may be taken to be Riemann integrals, and therefore the limit on the right is an improper Reimann integral.

Solution #2

This solution relies on the Abel summation formula,

$$\sum_{k=p}^{q} a_k b_k \;=\; \sum_{k=p}^{q-1} A_k \left(b_k - b_{k+1}\right) \;+\; A_q b_q \;-\; A_{p-1} b_p,\; r \leq p < q < \infty$$

where

$$A_s = \begin{cases} \sum\limits_{k=r}^{s} a_k, & s \geq r \\ 0, & s < r \end{cases} \tag{Abel}$$

Let $0 < a < b < \infty$. Restrict attention to the measurable set

$$U_{a,b} = f^{-1}\left([a,b)\right).$$

We shall prove that

$$\int\limits_{U_{a,b}} f^p\left(x\right) m\left(dx\right) = p \int\limits_{a}^{b} t^{p-1} e^{-t} dt + e^{-a} a^p - e^{-b} b^p, p > 0. \tag{2}$$

The monotone convergence theorem implies that if $a \to 0, b \to \infty$ through sequences, the left side tends to the left side of (1), and the right side tends to the right side of (1). (While the right side will be computed in the Riemann sense, the fact that the Riemann integral and Lebesgue integral of a Riemann integrable function are equal allow application of monotone convergence on the right.) Now for each $n > 0$, partition $U_{a,b}$ into measurable sets

$$U_{a,b}^{(n)}\left(k\right) = f^{-1}\left(\left[a + \tfrac{k-1}{n}\left(b-a\right), a + \tfrac{k}{n}\left(b-a\right)\right)\right)$$

$$m\left(U_{a,b}^{(n)}\left(k\right)\right) = \left(e^{-\left(a+\frac{k-1}{n}(b-a)\right)} - e^{-\left(a+\frac{k}{n}(b-a)\right)}\right), 1 \leq k \leq n.$$

The integral on the left of (2) is well approximated by the sum

$$\begin{aligned}
&\sum_{k=1}^{n}\left(a+\tfrac{k}{n}(b-a)\right)^{p} m\left(U_{a,b}^{(n)}(k)\right) \\
&=\sum_{k=1}^{n}\left(a+\tfrac{k}{n}(b-a)\right)^{p}\left(e^{-\left(a+\frac{k-1}{n}(b-a)\right)}-e^{-\left(a+\frac{k}{n}(b-a)\right)}\right).
\end{aligned} \tag{3}$$

Let $a_k = \left(e^{-\left(a+\frac{k-1}{n}(b-a)\right)}-e^{-\left(a+\frac{k}{n}(b-a)\right)}\right)$ and $b_k = \left(a+\frac{k}{n}(b-a)\right)^{p}$. Apply (Abel) to rewrite (3) as

$$\begin{aligned}
&\sum_{k=1}^{n-1}\left(e^{-a}-e^{-\left(a+\frac{k}{n}(b-a)\right)}\right)\left(\left(a+\tfrac{k}{n}(b-a)\right)^{p}-\left(a+\tfrac{(k+1)}{n}(b-a)\right)^{p}\right)+ \\
&\qquad\qquad +\left(e^{-a}-e^{-b}\right)b^{p} \\
&= e^{-a}\left(\left(a+\tfrac{1}{n}(b-a)\right)^{p}-b^{p}\right)+\left(e^{-a}-e^{-b}\right)b^{p} \\
&\quad+\sum_{k=1}^{n-1}e^{-\left(a+\frac{k}{n}(b-a)\right)}\left(\left(a+\tfrac{(k+1)}{n}(b-a)\right)^{p}-\left(a+\tfrac{k}{n}(b-a)\right)^{p}\right).
\end{aligned} \tag{4}$$

By the mean value theorem, there exists $\xi_{k,n} \in \left(a+\frac{k}{n}(b-a), a+\frac{(k+1)}{n}(b-a)\right)$ such that

$$\left(a+\frac{k}{n}(b-a)\right)^{p}-\left(a+\frac{(k+1)}{n}(b-a)\right)^{p}=p\xi_{k,n}^{p-1}\frac{(b-a)}{n}$$

Therefore, the sum on the last line of (4) is an approximating sum for the Riemann integral of the function $pt^{p-1}e^{-t}$. Letting $n \to \infty$ the extra terms approach $e^{-a}a^{p}-e^{-b}b^{p}$. Were one to use the method of solution #1, one would look at

$$\int\limits_{U_{a,b}} f^p(x)\, m(dx) = \int\limits_{[0,1]} \chi_{[a,b]}(f(x)) \left(\int\limits_0^{f(x)} pt^{p-1} dt \right) m(dx)$$

$$= \int\limits_{[0,1]} \chi_{[a,b]}(f(x)) \left(\int\limits_0^{\infty} pt^{p-1} \chi_{[0,f(x)]}(t)\, dt \right) m(dx)$$

$$= \int\limits_{[0,1]} \chi_{[a,b]}(f(x)) \left(\int\limits_0^{\infty} pt^{p-1} \chi_{[t,\infty)}(f(x))\, dt \right) m(dx)$$

$$\overset{\text{Fubini}}{=} \int\limits_0^{\infty} pt^{p-1} \left(\int\limits_{[0,1]} \chi_{[a,b]}(f(x))\, \chi_{[t,\infty)}(f(x))\, m(dx) \right) dt$$

$$= \int\limits_0^{a} pt^{p-1} m(U_{a,b})\, dt + \int\limits_a^{b} pt^{p-1} \left(\int\limits_{[0,1]} m(\{x | t \le f(x) \le b\}) \right) dt$$

$$\overset{\text{Fubini}}{=} a^p \left(e^{-a} - e^{-b}\right) + \int\limits_a^{b} pt^{p-1} e^{-t} dt - e^{-b}\left(b^p - a^p\right)$$

$$= \int\limits_a^{b} pt^{p-1} e^{-t} dt + e^{-a} a^p - e^{-b} b^p.$$

4.3. (Problem 4, continued.) Use (1), even if you were unable to prove it, to establish the bound

$$\left| \int\limits_{[0,1]} f^p(x)\, m(dx) - \frac{1}{n} \sum_{k=1}^{\infty} e^{-\left(\frac{k}{n}\right)^{\frac{1}{p}}} \right| \le \frac{1}{n}, n > 0.$$

(Hint: Consideration of $F_p(x) = f^p(x)$ and the case $p = 1$ of (1) for $F_p(x)$ might help.)

Solution

Let $F_p(x) = f(x)^p$. Then

$$m(\{x|F_p(x) \geq \tau\}) = m\left(\left\{x|f(x) \geq \tau^{\frac{1}{p}}\right\}\right) = e^{-\tau^{\frac{1}{p}}}.$$

One has

$$\int_{[0,1]} f^p(x)\, m(dx) = \int_{[0,1]} F_p(x)\, m(dx)$$

$$= \int_0^\infty e^{-\tau^{\frac{1}{p}}}\, dt.$$

Given $n > 1$, partition $[0, \infty)$ into intervals $[\frac{k-1}{n}, \frac{k}{n}), k \geq 1$. We have

$$\frac{1}{n}\sum_{k=1}^{\infty} e^{-\left(\frac{k}{n}\right)^{\frac{1}{p}}} \leq \int_{[0,1]} f^p(x)\, m(dx)$$

$$\leq \frac{1}{n}\sum_{k=1}^{\infty} e^{-\left(\frac{k-1}{n}\right)^{\frac{1}{p}}}.$$

The sums are identical but for the extra summand, which is 1, on the right. Therefore, either sum approximates the integral to within $\frac{1}{n}$.

Problem 5

This problem has two parts, one page per part. Let $f_n \geq 0$ be Lebesgue measurable on $\mathbb{R}$ for all $n \geq 1$, and assume

$$\int_{\mathbb{R}} f_n(x)\, dx = 100, n \geq 1$$

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ exists a.e. } x.$$

5.1. Is f integrable? If your answer is 'yes', cite a theorem in support of it. If it is 'no', explain why there is a counterexample.

Solution

Yes. Direct application of Fatou's Lemma implies

$$\int_{\mathbb{R}} f(x)\,dx \le 100.$$

5.2. (Problem 5, continued.) Assume $\int_{\mathbb{R}} f(x)\,dx = 31$. Do there exist $T < \infty$ and n_0 such that $\int_{[-T,T]} f_n(x)\,dx > 30.9, n \ge n_0$? If your answer is 'yes', cite theorem(s) in support of it. If it is 'no', explain why there is a counterexample.

Solution

Since f is nonnegative, the monotone convergence theorem implies that

$$\lim_{L\to\infty} \int_{[-L,L]} f(x)\,dx = 31.$$

Select L so that

$$\int_{[-L,L]} f(x)\,dx = 30.9 + \delta, \delta > 0.$$

Fatou's Lemma implies that

$$\liminf_{n\to\infty} \int_{[-L,L]} f_n(x)\,dx \ge \int_{[-L,L]} f(x)\,dx = 30.9 + \delta.$$

Therefore, there exists n_0 such that

$$\int_{[-L,L]} f_n(x)\,dx > 30.9, n \geq n_0$$

Problem 6

NOTATIONS and ASSUMPTIONS Let $\Lambda \subset (0,1)$ denote the set of points such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^3}$$

admits an **infinite number** of reduced rational solutions $\frac{p}{q} \in (0,1)$, i.e., with $q > 0$ and p, q relatively prime with $1 \leq p < q$. This problem has three parts, one page per part.

6.1. For each pair (p,q) define an appropriate interval $J(p,q)$ and by carefully filling in the ranges of n, p, q below, represent the set Λ as

$$\Lambda = \bigcap_n \left(\bigcup_q \left(\bigcup_p J(p,q) \right) \right).$$

Solution

Let $\frac{p}{q} \in (0,1)$, with p, q relatively prime. Define $J(p,q)$ to be the open interval

$$J(p,q) = \left\{ x \in (0,1) \mid \left| x - \frac{p}{q} \right| < \frac{1}{q^3} \right\}.$$

Now, Λ is the (G_δ) set

$$\Lambda = \bigcap_{n=1}^{\infty} \left(\bigcup_{q \geq n} \left(\bigcup_{(p,q)=1, p<q} J(p,q) \right) \right).$$

6.2. (Problem 6. continued.) Can you say that Λ is a Borel set? Why? Your answer should include the definition of 'Borel set.

Solution

The Borel sets are the sets in the σ-algebra generated by the open sets. $\left(\bigcup_{q\geq n}\left(\bigcup_{(p,q)=1,p<q} J(p,q)\right)\right)$ is an open set, and therefore Λ is a Borel set, even a G_δ.

6.3. (Problem 6. continued.) What can you say about the Lebesgue measure of Λ? Explain your answer.

Solution

The number of $p < q$ such that $(p,q) = 1$ is bounded by q. Therefore the measure of $\bigcup_{(p,q)=1,p<q} J(p,q)$ is bounded by $q \cdot \frac{1}{q^3} = \frac{1}{q^2}$. Therefore, by the integral test,

$$m\left(\bigcup_{q\geq n}\left(\bigcup_{(p,q)=1,p<q} J(p,q)\right)\right) \leq \sum_{q\geq n}\frac{1}{q^2} = o(1).$$

By definition, $\Lambda \subseteq \bigcup_{q\geq n}\left(\bigcup_{(p,q)=1,p<q} J(p,q)\right)$, and therefore, $m(\Lambda) = 0$.