August 2010 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 3 hour, closed book, closed notes exam. Show your work. Problems with multiple parts may carry more credit than those without multiple parts.

1. Prove that for $n \neq 1$, any continuous map $f : \mathbb{RP}(n) \times \mathbb{RP}(n) \to S^1 \times S^1$ is null-homotopic.

2. Describe a topological space X with $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, $\pi_1(X) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5$, $H_2(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_6$, $H_6(X;\mathbb{Z}) \cong \mathbb{Z}$ and all other homology groups zero. Justify your answer.

3. Calculate the integral cohomology groups of the space in Problem 2 above. Then prove that any continuous map $f : X \to \mathbb{C}P(3)$ induces the zero map on $H_6(-;\mathbb{Z})$.

4. (a) Suppose Y is a compact, orientable n-dimensional manifold (possibly with boundary). Prove that $H_{n-1}(Y;\mathbb{Z})$ is torsion-free.
   (b) Let $f : H \to S^3$ be a topological embedding where H is a solid handlebody of genus 2 (i.e. a thickened wedge of circles) as in Figure 1. Let $\text{int}(f(H))$ be the interior of the image of f in S^3. Compute $H_p(S^3 - \text{int}(f(H)))$ for all p.

   Hint: $S^3 = (S^3 - \text{int}(f(H))) \cup f(H)$.

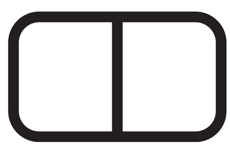

FIGURE 1. H, a solid handlebody of genus 2

5. Suppose Γ is a finitely generated group. Prove that there exists a 1-dimensional CW-complex on which Γ acts freely and properly discontinuously. What is the quotient (orbit) space in your example?

6. Suppose $A \subset X$ is a CW pair. State and sketch a proof of the existence of the long exact **cohomology** sequence for the pair (X, A). Hint: Zig-Zag-Lemma.