This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Prove that $SL_n(\mathbb{R})$ is a smooth manifold and calculate its dimension (with proof).

2. Let $F(n)$ be the free group of rank $n$. For each integer $n \geq 2$, prove that $F(2)$ contains a finite index normal subgroup isomorphic to $F(n)$.

3. Let $W = S^1 \vee S^1 \vee S^1$ be as shown below. Let $x, y, z$ be the three loops indicated going around the first, second, third circles respectively. Let $X = W \cup f_1 e_1^1 \cup f_2 e_2^1$ be the space obtained from $W$ by adjoining one 2-cell via the map $f_1$ which forms the loop $x y x^{-1} y^{-1} z^{-1}$; and another 2-cell via the map $f_2$ which forms the loop $x^7$.

(a) Write down a cellular chain complex for $X$ (including the boundary maps).

(b) Use (a) to compute $H_p(X; \mathbb{Z}_2)$ and $H_p(X; \mathbb{Z})$ for all $p$. Do not use a Universal Coefficient Theorem.

(c) Use the Universal Coefficient Theorem for Cohomology to compute $H^p(X; \mathbb{Q})$ for all $p$.

4. Let $T = S^1 \times S^1$ and let $f : T \to T$ be defined by

$$f(x, y) = (2x + y, x + y).$$

Here we are viewing $S^1$ as $\mathbb{R}/\mathbb{Z}$. Let $X = (T \times [0, 1]) / \sim$ be the 3-manifold obtained by identifying $(x, y) \times \{0\}$ with $f(x, y) \times \{1\}$. Compute $\pi_1(X)$. 

1
5. Let $M$ be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_5 \ast \mathbb{Z}_5$ and $\chi(M) = 5$.

(a) Compute $H_p(M;\mathbb{Z})$ for all $p$.

(b) Prove that $M$ is not homotopy equivalent to a CW complex with no 3-cells.

6. Let $n \geq 0$ be an even integer. Prove that there is no orientation-reversing (that is degree $-1$) map $f : \mathbb{C}P^n \to \mathbb{C}P^n$. 