RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2016

This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let M be a smooth, closed (compact and boundaryless), connected m-dimensional manifold and S^n be the standard unit sphere in $\mathbb{R}^{n+1}$ (with its standard smooth structure). Suppose $f : M \to S^n$ is a smooth map and $0 \leq m < n$. Prove that f is nullhomotopic.

2. Let $X = S^1 \times \mathbb{R}P^2$, where $\mathbb{R}P^2$ is the real projective plane. Discuss all of the covering spaces of X. How many are there? Describe them. What are their groups of covering transformations? Which ones are regular (normal)?

3. Suppose S is a compact, connected surface without boundary. Suppose $\pi_1(S)$ has a *proper* finite index subgroup that is isomorphic to $\pi_1(S)$. For which surfaces is this possible? (Prove it)

4. Let X be the space obtained from a solid octagon by identifying sides as shown Figure 1 below.

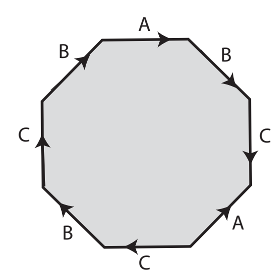

FIGURE 1

   (a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex.
   (b) Give a presentation for $\pi_1(X)$.
   (c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{Q})$ for all $n \geq 0$.
   (d) Prove or disprove: X has the homotopy type of a closed m-dimensional manifold for some $m \geq 0$.

5. Suppose M is a compact, connected, orientable 3-dimensional manifold with non-empty boundary ∂M. If $\pi_1(M)$ is finite, prove that ∂M is a disjoint union of 2-spheres. (Hint: Calculate $H_1(\partial M; \mathbb{Q})$.)

6. Let M be a closed, connected, oriented 4-dimensional manifold with $\beta_2(M) \neq 0$. Prove that any continuous map $f : S^4 \to M$ has degree equal to 0.