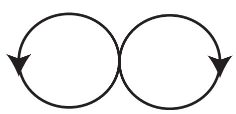
January 2013 - TOPOLOGY QUALIFYING EXAM - RICE UNIVERSITY

This is a 3 hour, closed book, closed notes exam. Please include thorough justifications, as much as time allows. Sign the honor pledge at the conclusion of the exam.

(1) If X is a 2-dimensional disk with two open sub-disks removed, discuss **2-fold** covering spaces of X. Your discussion might include: relations to $\pi_1(X)$, examples, pictures, what are the possible groups of covering transformations, how many covers are there up to equivalence, are they regular?

(2) Determine exactly which cyclic groups can act (freely and) properly discontinuously on the closed orientable surface, Σ_8, of genus 8. Which of these can act on Σ_8 with orientable quotient?

(3) Let $X = S^1 \vee S^1$ be a wedge of circles labelled x and y (as indicated below). Let Y be the space obtained by attaching two 2-cells to X along the curves $x^2y^{-1}xy^2$ and yx^5yx, respectively.



(a) Compute a presentation of $\pi_1(Y)$ and express its abelianization in the form $\mathbb{Z}^m \oplus \mathbb{Z}_{n_1} \oplus \ldots$.
(b) Compute $H_p(Y; \mathbb{Z}_3)$ for all p.
(c) Compute $H^p(Y; \mathbb{Z})$ for all p without using a Universal Coefficient Theorem.

(4) Let X be a closed, connected, orientable 4-dimensional manifold with $\pi_1(X) \cong \mathbb{Z}_{15}$ and $H_2(X; \mathbb{Q}) \cong \mathbb{Q}^3$. Let E be a connected 3-fold covering space of X.
(a) What is $\pi_1(E)$?
(b) Calculate $H_i(X; \mathbb{Z})$ for each i. What is $\chi(X)$?
(c) Calculate $H_i(E; \mathbb{Z})$ for each i.
(d) Prove that E admits no CW decomposition without 3-cells.

(5) Let $f : S^2 \times S^6 \to \mathbb{C}P^4$ be a continuous map. Prove that f induces the zero map on $H_6(-; \mathbb{Z})$ and $H_8(-; \mathbb{Z})$.

(6) Suppose (X, A) is a pair of topological spaces. State, and sketch the proof of, the theorem asserting the existence of a long exact sequence in homology for this pair. The proof sketch is viewed as the major content of this problem. Define the boundary homomorphism (the one that decreases degree).