1. Suppose \( f: \mathbb{R}^n \to \mathbb{R}^m \) is a smooth map and \( n > m \). Prove that \( f \) cannot be one-to-one.

2. Let \( F(n) \) be the free group of rank \( n \). For each integer \( n \geq 2 \), prove that \( F(2) \) contains a finite index normal subgroup isomorphic to \( F(n) \).

3. Let \( f: H \to \mathbb{R}^3 \) be a topological embedding where \( H \) is a solid handlebody of genus 2 (i.e. a thickened \( \theta \) graph) as in Figure 1. Let \( X = \text{int}(f(H)) \) be the image of \( f \) in \( \mathbb{R}^3 \). Compute \( H_p(\mathbb{R}^3 - X) \) for all \( p \).

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure1.png}
\caption{\( H \), a solid handlebody of genus 2}
\end{figure}

4. Let \( T = S^1 \times S^1 \) and let \( f: T \to T \) be defined by
\[
f(x, y) = (2x + y, x + y).
\]
Here we are viewing \( S^1 \) as \( \mathbb{R}/\mathbb{Z} \). Let \( X = (T \times [0, 1])/\sim \) be the 3-manifold obtain by identifying \( (x, y) \times \{0\} \) with \( f(x, y) \times \{1\} \). Compute \( \pi_1(X) \).

5. Let \( M \) be a closed, connected, orientable 4-dimensional manifold with \( \pi_1(M) \cong \mathbb{Z}_5 \ast \mathbb{Z}_5 \) and \( \chi(M) = 5 \).
   (a) Compute \( H_p(M; \mathbb{Z}) \) for all \( p \).
   (b) Prove that \( M \) is not homotopy equivalent to a CW complex with no 3-cells.

6. Let \( n \geq 1 \). Prove that there is no orientation-reversing (that is degree \(-1\)) map \( f: \mathbb{C}P(2n) \to \mathbb{C}P(2n) \).