ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2019

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 22nd, 2019.

(1) Insert group theory problem here.

(2) Are any of the rings $\mathbb{F}_2[x]/(x^2+x+1)$, $\mathbb{Z}[\sqrt{-2}]/(2)$ and $\mathbb{Z}[\sqrt{-3}]/(1+\sqrt{-3})$ isomorphic to one another? Justify carefully.

(3) Insert linear algebra problem here.

(4) Let $a = \cos(2\pi/9)$.

(a) Compute the minimal polynomial $P(x)$ of a over $\mathbb{Q}$.

(b) Is $\mathbb{Q}(a)/\mathbb{Q}$ separable? Is it a splitting field for $P(x)$?

(c) Compute $\mathrm{Aut}(\mathbb{Q}(a)/\mathbb{Q})$.

Carefully justify your answers.

(5) For a commutative ring with unit A, let $\mathcal{J}(A)$ denote its Jacobson radical.

(a) Let A and B be commutative rings with unit. Show that

$$\mathcal{J}(A \oplus B) = J(A) \oplus J(B)$$

as ideals in $A \oplus B$.

(b) Let A be a commutative ring with unit, and let

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} : a, b, c \in A \right\},$$

which is itself a commutative ring with unit. Show that

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in \mathcal{J}(R)$$

if and only if a, $b \in \mathcal{J}(A)$.

(6) (a) Let $f\colon \mathbb{Q}[t] \to \mathbb{Q}[x,y]/(xy)$ be the ring homomorphism determined by $f(t) = x$. The map f gives $\mathbb{Q}[x,y]/(xy)$ a $\mathbb{Q}[t]$-module structure. Show that $\mathbb{Q}[x,y]/(xy)$ is *not* a finite $\mathbb{Q}[t]$-module.

(b) Must there exist a ring homomorphism $f\colon \mathbb{Q}[t] \to \mathbb{Q}[x,y]/(xy)$ such that $\mathbb{Q}[x,y]/(xy)$ is a finite $\mathbb{Q}[t]$-module? If so, prove it by writing down an explicit map and show this map is finite. If not, explain why no such f exists.