ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2021

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Honor pledge: *On my honor, I have neither given nor received any unauthorized aid on this exam.*

Date: August 16, 2021.

(1) Show that a group of order 600 cannot be simple.

(2) Prove that the polynomial $f(t) = t^4 + 4t^3 + 6t^2 + 2t + 1 \in \mathbb{Z}[t]$ is irreducible.

(3) Let p be an **odd** prime number, and let $\alpha \in \mathbb{F}_p$ be a **nonzero** element of a finite field with p elements. Let M be the 3×3 matrix over $\mathbb{F}_p$ given by

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Show that $M^p = I$.

(b) Is the matrix M diagonalizable? Justify your answer.

(4) (a) Let $M/L/K$ be a tower of field extensions such that M/K is Galois. Explain why L/K is a separable extension.

(b) Let K be a field, and let L/K and M/K be finite Galois extensions of K, both contained in a fixed algebraic closure of K. Prove that the field $L \cap M$ is Galois over K.

(5) (a) Show that there are only finitely many ideals in $\mathbb{R}[x, y]$ that contain the ideal

$$I := (x^2 + 1, y^2 + 1).$$

(b) Prove that the ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has finitely many prime ideals. How many exactly?

(6) Let R be a commutative ring with 1.

(a) Prove that the tensor product of two free R-modules is free.

(b) Prove that the tensor product of two projective R-modules is projective.