ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2021

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam: "On my honor, I have neither given nor received any unauthorized aid on this (assignment, exam, paper, etc.)."

Date: May 10, 2021.

(1) Prove that a group of order 99 is abelian.
(You may use the fact that, for p, q prime numbers, groups of order p, p^2, or pq are abelian.)

(2) Let R be a commutative ring. Prove that the polynomial ring $R[x]$ is a principal ideal domain if and only if R is a field.

(3) Let R be a ring, and let X and Y be finite sets of the same cardinality. Prove that the free R-modules $F(X)$ and $F(Y)$ on the respective sets X and Y are isomorphic as R-modules, via universal property.

(4) Construct, with proof, a Galois extension K of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$.

(5) Prove that the integral closure of $\mathbb{Z}$ in the field $\mathbb{Q}(i)$ is the ring of Gaussian integers $\mathbb{Z}[i]$.

(6) Let R be a commutative ring with unit, and let M be an R-module. Suppose that $\{f_i\}$ is a set of elements that generates the unit ideal of R.

   (a) Show that for any integer $N > 1$ the elements $\{f_i^N\}$ also generate the unit ideal in R.

   (b) Prove that if $m \in M$ maps to zero under all of the localization maps $M \to M_{f_i}$, then $m = 0$.