ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2020

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 13th, 2020.

(1) An *automorphism* α of a group G is an isomorphism $\alpha\colon G \to G$. A subgroup $H < G$ is called *characteristic* if $\alpha(H) = H$ for all automorphisms α of G.

(a) Prove that if H is a characteristic subgroup of G then H is a normal subgroup of G.

(b) Let K be a normal subgroup of G and M a characteristic subgroup of K. Prove that M is normal in G.

(2) Let $\phi\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the map of abelian groups associated to the matrix

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{pmatrix}.$$

Is the cokernel of the ϕ a finite abelian group? Justify your answer.

(3) Let $M_2(\mathbb{F}_5)$ denote the vector space of 2×2 matrices with entries in the finite field $\mathbb{F}_5$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $T\colon M_2(\mathbb{F}_5) \to M_2(\mathbb{F}_5)$ be the linear transformation defined by

$$T(X) = XA - AX.$$

Determine the minimal polynomial of T, as well as a rational canonical form for this linear transformation.

(4) Let $\zeta = e^{2\pi i/14} \in \mathbb{C}$ be a primitive 14-th root of unity, and let $K = \mathbb{Q}(\zeta)$.

(a) Is the group $\mathrm{Gal}(K/\mathbb{Q})$ abelian?

(b) How many intermediate extensions $\mathbb{Q} \subseteq L \subseteq K$ are there?

Justify your answers.

(5) Let M be a module over an integral domain R such that the annihilator ideal

$$\mathrm{Ann}(M) = \{a \in R : aM = 0\} \subset R$$

is nonzero. Show that M cannot be flat.

(6) Let R be a valuation ring, i.e., an integral domain with field of fractions K such that for every $x \in K$, either $x \in R$ or $x^{-1} \in R$. Prove that R is integrally closed.