ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2021

Instructions:

• You should complete this exam in a single **four** block of time. Attempt all **six** problems.
• The use of books, notes, calculators, or other aids is **not** permitted.
• Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
• Write and sign the Honor Code pledge at the end of your exam.
(1) Suppose that the cyclic group $C_7$ of order 7 acts on the $2 \times 2$ matrix ring $\text{Mat}_2(\mathbb{Z}_2)$ over the finite field with two elements $\mathbb{Z}_2$. Are there matrices in $\text{Mat}_2(\mathbb{Z}_2)$ that remain fixed under such a group action? Explain why or why not.

(2) Consider the ring of Gaussian integers $\mathbb{Z}[i]$, for $i := \sqrt{-1}$. Let $J$ be any nonzero ideal of $\mathbb{Z}[i]$. Prove that the quotient ring $\mathbb{Z}[i]/J$ has finite cardinality as a set.

(3) Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

(4) List similarity class representatives for $5 \times 5$ complex matrices $A$ that satisfy the equation $A^5 = A^3$.

(5) Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$ be a primitive $n$-th root of unity. Does there exist an $n$ such that $\mathbb{Q}(\sqrt[3]{19}) \subseteq \mathbb{Q}(\zeta_n)$? If so, find one (with proof); if not, show why not.

(6) Let $A$ be an integral domain. Show that

$$A = \bigcap_m A_m,$$

where the intersection runs over all maximal ideals $m$ of $A$.

[Hint: For $a \in \bigcap_m A_m$, consider the ideal $I_a := \{x \in A : xa \in A\}$ of $A$. What does $a \in A$ imply for the ideal $I_a$?]