ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2021

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 26, 2021.

(1) Suppose that the cyclic group C_7 of order 7 acts on the 2×2 matrix ring $\mathrm{Mat}_2(\mathbb{Z}_2)$ over the finite field with two elements $\mathbb{Z}_2$. Are there matrices in $\mathrm{Mat}_2(\mathbb{Z}_2)$ that remain fixed under such a group action? Explain why or why not.

(2) Consider the ring of Gaussian integers $\mathbb{Z}[i]$, for $i := \sqrt{-1}$. Let J be any nonzero ideal of $\mathbb{Z}[i]$. Prove that the quotient ring $\mathbb{Z}[i]/J$ has finite cardinality as a set.

(3) Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

(4) List similarity class representatives for 5×5 complex matrices A that satisfy the equation $A^5 = A^3$.

(5) Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$ be a primitive n-th rooth of unity. Does there exist an n such that $\mathbb{Q}(\sqrt[3]{19}) \subseteq \mathbb{Q}(\zeta_n)$? If so, find one (with proof); if not, show why not.

(6) Let A be an integral domain. Show that

$$A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}},$$

where the intersection runs over all maximal ideals $\mathfrak{m}$ of A.

[Hint: For $a \in \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$, consider the ideal $I_a := \{x \in A : xa \in A\}$ of A. What does $a \in A$ imply for the ideal I_a?]