Analysis Exam, May 2020

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Denote $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
   (a) Let $f : \mathbb{C} \to \mathbb{C}_+$ be an analytic function. Prove that f is constant.
   (b) Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}_+$ be an analytic function. Prove that f is constant.

2. (a) Let $f \in L^1([0, 2\pi], dx)$ and define $\hat{f}_n = \int_0^{2\pi} e^{-inx} f(x)\, dx$ for $n \in \mathbb{Z}$. Prove that $\lim_{n\to\infty} \hat{f}_n = 0$.
   (b) Prove that there exists a finite positive measure μ on $[0, 2\pi]$ such that the Fourier coefficients $\hat{\mu}_n = \int_0^{2\pi} e^{-inx}\, d\mu(x)$ do not converge to 0 as $n \to \infty$.

3. Prove that the integral

$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx$$

   exists and find its value.

4. (a) Define the total variation of a function $f : [0, 1] \to \mathbb{C}$.
   (b) Assuming that f has finite total variation, estimate the total variation of the function $g : [0, 1] \to \mathbb{C}$, $g(x) = \int_0^1 f(xy)\, dy$, in terms of the total variation of f.
   (c) If f is absolutely continuous, prove that g is absolutely continuous.

5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are integrable. Define the *convolution* of f and g, denoted by $f * g$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x - y)dy.$$

   1. If f and g are in L^1, then prove that $f * g$ is in L^1.
   2. How are $(f * g)(x)$ and $(g * f)(x)$ related?
   3. Give an example of a smooth L^1 function f and a discontinuous L^1 function g with the property that $f * g$ is smooth.
   4. Give an example of a discontinuous L^1 function f and a discontinuous L^1 function g with the property that $f * g$ is continuous.

6. Suppose that f is analytic on the disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $\epsilon > 0$, and $\lim_{n\to\infty} f(z_n) = 0$ for any sequence $z_n \in \mathbb{D}$ that converges to $e^{i\theta}$ for some $\theta \in (0, \epsilon)$. Prove that f is identically 0.