Analysis Exam, May 2020

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Denote \( \mathbb{C}_+ = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} \).
   
   (a) Let \( f : \mathbb{C} \to \mathbb{C}_+ \) be an analytic function. Prove that \( f \) is constant.
   
   (b) Let \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C}_+ \) be an analytic function. Prove that \( f \) is constant.

2. (a) Let \( f \in L^1([0, 2\pi], dx) \) and define \( \hat{f}_n = \int_0^{2\pi} e^{-inx} f(x) dx \) for \( n \in \mathbb{Z} \). Prove that \( \lim_{n \to \infty} \hat{f}_n = 0 \).

   (b) Prove that there exists a finite positive measure \( \mu \) on \([0, 2\pi]\) such that the Fourier coefficients \( \hat{\mu}_n = \int_0^{2\pi} e^{-inx} d\mu(x) \) do not converge to 0 as \( n \to \infty \).

3. Prove that the integral
   \[
   \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx
   \]
   exists and find its value.

4. (a) Define the total variation of a function \( f : [0, 1] \to \mathbb{C} \).

   (b) Assuming that \( f \) has finite total variation, estimate the total variation of the function \( g : [0, 1] \to \mathbb{C}, g(x) = \int_0^1 f(xy) \, dy \), in terms of the total variation of \( f \).

   (c) If \( f \) is absolutely continuous, prove that \( g \) is absolutely continuous.

5. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) are integrable. Define the convolution of \( f \) and \( g \), denoted by \( f \ast g \), by
   \[
   f \ast g(x) = \int_{\mathbb{R}^n} f(y)g(x - y)dy.
   \]

   1. If \( f \) and \( g \) are in \( L^1 \), then prove that \( f \ast g \) is in \( L^1 \).

   2. How are \((f \ast g)(x)\) and \((g \ast f)(x)\) related?

   3. Give an example of a smooth \( L^1 \) function \( f \) and a discontinuous \( L^1 \) function \( g \) with the property that \( f \ast g \) is smooth.

   4. Give an example of a discontinuous \( L^1 \) function \( f \) and a discontinuous \( L^1 \) function \( g \) with the property that \( f \ast g \) is continuous.

6. Suppose that \( f \) is analytic on the disk \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \), \( \epsilon > 0 \), and \( \lim_{n \to \infty} f(z_n) = 0 \) for any sequence \( z_n \in \mathbb{D} \) that converges to \( e^{i\theta} \) for some \( \theta \in (0, \epsilon) \). Prove that \( f \) is identically 0.