1. Let $K$ be a compact metric space. Let $f_n : K \to \mathbb{C}$ be continuous functions for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and let $f : K \to \mathbb{C}$ be a continuous function.

Prove that the functions $f_n$ converge to $f$ uniformly on $K$ as $n \to \infty$ if and only if the function $g : K \times (\{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}) \to \mathbb{C}$ defined by

$$g\left(x, \frac{1}{n}\right) = f_n(x), \quad g(x, 0) = f(x)$$

is a continuous function on $K \times (\{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\})$.

2. For $-1 < a < 2$, compute the improper integral

$$\int_0^{+\infty} \frac{x^a}{1 + x^3} \, dx$$

3. Consider the vertical strip $\{z \in \mathbb{C} | \text{Re} \, z \in (0, 1)\}$ in the complex plane. Find all functions $f(z)$ which are holomorphic in the strip and continuous on the closed strip $\{z \in \mathbb{C} | \text{Re} \, z \in [0, 1]\}$, purely imaginary on the boundary edge $x = 0$, have Re $f(z) = 7$ on the boundary edge $x = 1$, and which satisfy $|f(z)| \leq A + |z|^n$ for some fixed positive integer $n$ and some positive real number $A > 0$.

4. Let $p(z)$ be a polynomial. Suppose the zeroes of $p(z)$ lie in the right half plane $\{\text{Re} \, z > 0\}$. Show that the zeroes of the derivative $p'(z)$ also lie inside right half plane.

*Hint: consider $\frac{p'(z)}{p(z)}$ and write this out as a sum of functions with simple poles at the zeroes of $p(z)$. Rewrite each term as a fraction with a real denominator and then study the expression at a zero of $p'(z)$.*

5. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of positive measurable functions on a measure space $(X, \mathcal{B}, \mu)$. Suppose that the sum

$$\sum_{k=1}^{\infty} \mu\{x \in X | f_k(x) \geq \epsilon\}$$

converges for every $\epsilon > 0$. Prove that $f_k(x) \to 0$ almost everywhere on $X$.

6. Let $\{f_n\}$ be a sequence in $L^2([0, 1])$ with uniformly bounded $L^2$ norm: $\|f_n\| < 17$. Show that if $f_n$ converges to zero in measure (with respect to Lebesgue measure), then $f_n$ converges to zero in $L^1$.

*Recall that “$f_n$ converges to zero in measure” if, for every $\epsilon > 0$, we have $\lim_{n \to \infty} \mu(\{|f_n(x)| > \epsilon\}) = 0$.**