Analysis Exam, May 2022

Please put your name on your solutions, use different 8 1/2×11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Assume that $f(z)$ is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If $f(z) = f(1/z)$ when $|z| = 1$, prove that $f(z)$ is constant.

2. Let s_n be a sequence of complex numbers such that $\lim_{n\to\infty} s_n = s \in \mathbb{C}$. For $x \in (0,1)$ define

$$f(x) = \sum_{n=0}^{\infty} s_n x^n (1-x).$$

   Prove that $\lim_{x\to 1} f(x) = s$.

3. Find the set of analytic bijections $f : \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$.

4. Assume $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

   Prove that $f = 0$ Lebesgue-almost everywhere.

   Hint: find the derivative of the function $g(x) = \int_{-\infty}^{x} f(t)\, dt$.

5. Evaluate the integral

$$\int_0^\infty \frac{\sin(ax)}{x(1+x^2)}\, dx$$

   where $a > 0$.

6. Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that for some $0 < p < 1$, we have

$$\left| \int f(x) g(x) dx \right| \le \left(\int |g(x)|^p dx \right)^{1/p},$$

   for all $g \in C_0(\mathbb{R}^d)$ (continuous functions with compact support). Show that $f(x) = 0$ almost everywhere.