Analysis Exam, August 2023

Please put your name on your solutions, use different 8 1/2 × 11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Let \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \), and suppose that \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic, \( f(1/3) = 0 \) and \( f'(1/3) = 0 \). Show that \( |f(0)| \leq 1/9 \).

2. For \( r > 0 \), let \( B_r \) denote the radius \( r \) open ball in \( \mathbb{R}^n \) centered at the origin. Let \( f : B_2 \to \mathbb{R} \) be \( C^3 \). Prove that there exists a \( C > 0 \) such that

\[
|f(x) - f(y)| \leq C|x - y|^3
\]

for all \( x, y \in B_1 \) satisfying \( \nabla f(x) = \nabla f(y) = 0 \).

3. Find the value of the integral

\[
\int_{-\infty}^{+\infty} \frac{\log(1 + x^2)}{4 + x^2} \, dx.
\]

4. (a) Suppose that \( f \in L^2(\mathbb{R}) \) with the Lebesgue measure, and \( x_n \) is a sequence of real numbers so that \( x_n \to 0 \). Define \( f_n(x) = f(x + x_n) \). Show that \( f_n \) converges to \( f \) in \( L^2 \).

(b) Let \( X \) be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences \( X - X = \{ x - y : x, y \in X \} \subset \mathbb{R} \) contains an open neighborhood of the origin.

5. Denote \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and consider a sequence of holomorphic functions \( f_n : \mathbb{D} \to \mathbb{D} \). Assume that \( f_n(0) \to 0 \) and \( |f_n(1/2)| \to 1/2 \) as \( n \to \infty \). Prove that for each \( w \in \mathbb{D} \), there exists \( n_0 \) such that \( w \) is in the range of \( f_n \) for all \( n \geq n_0 \).

6. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a measurable function satisfying \( f(x + 1) = f(x) \) and \( f(2x) = f(x) \) for almost every \( x \in \mathbb{R} \). Show that there exists a constant \( c \in \mathbb{R} \) such that \( f(x) = c \) for almost every \( x \in \mathbb{R} \).