Analysis Exam, August 2023

Please put your name on your solutions, use different 8 1/2×11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, and suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, $f(1/3) = 0$ and $f'(1/3) = 0$. Show that $|f(0)| \leq 1/9$.

2. For $r > 0$, let B_r denote the radius r open ball in $\mathbb{R}^n$ centered at the origin. Let $f : B_2 \to \mathbb{R}$ be C^3. Prove that there exists a $C > 0$ such that
$$|f(x) - f(y)| \leq C\|x - y\|^3$$
for all $x, y \in B_1$ satisfying $\nabla f(x) = \nabla f(y) = 0$.

3. Find the value of the integral
$$\int_{-\infty}^{+\infty} \frac{\log(1 + x^2)}{4 + x^2}\, dx.$$

4. (a) Suppose that $f \in L^2(\mathbb{R})$ with the Lebesgue measure, and x_n is a sequence of real numbers so that $x_n \to 0$. Define $f_n(x) = f(x + x_n)$. Show that f_n converges to f in L^2.
   (b) Let X be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences $X - X = \{x - y : x, y \in X\} \subset \mathbb{R}$ contains an open neighborhood of the origin.

5. Denote $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and consider a sequence of holomorphic functions $f_n : \mathbb{D} \to \mathbb{D}$. Assume that $f_n(0) \to 0$ and $|f_n(1/2)| \to 1/2$ as $n \to \infty$. Prove that for each $w \in \mathbb{D}$, there exists n_0 such that w is in the range of f_n for all $n \geq n_0$.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying $f(x+1) = f(x)$ and $f(2x) = f(x)$ for almost every $x \in \mathbb{R}$. Show that there exists a constant $c \in \mathbb{R}$ such that $f(x) = c$ for almost every $x \in \mathbb{R}$.