Rice University Algebra Qualifying Exam Syllabus

(updated April 2021)

Group Theory.

1. Basic theory: subgroups, normal subgroups, cosets, quotients, isomorphism theorems, conjugacy classes, direct products, semidirect products, commutator subgroup, solvable and nilpotent groups.

2. Examples: abelian groups, cyclic groups, dihedral groups, symmetric groups, alternating groups, finite matrix groups.

3. Group actions: definition, orbits, stabilizers, orbit-stabilizer theorem, conjugation action, action on cosets, symmetry groups of simple geometric objects (e.g., regular polygons).

4. Generators and relations: free groups, definition of group presentation, basic examples (e.g., surface groups).

5. Finite group theory: Sylow's theorems, classification of groups of small order.

References: A standard reference for this is Dummit–Foote's *Abstract Algebra*, Chapters 1–6. A somewhat easier reference that covers some of it is Artin's *Algebra*, Chapters 2,6,7; however, this does not include everything. A more advanced but still very readable source is Alperin–Bell's *Groups and Representations*.

Ring and Module Theory.

1. Ring Theory: ideals, homomorphisms, quotient rings and their ideals, product rings. Maximal and prime ideals.

2. Polynomial rings: division algorithm, Gröbner bases and applications to the ideal membership problem and elimination theory. Hilbert Basis Theorem.

3. Factorization: irreducible vs prime elements, Euclidean domains, principal ideal domains and unique factorization domains. Gauss' Lemma; factorization of polynomials in \Z[x] (Eisenstein's criterion, reduction mod p).

4. Module Theory: submodules, quotient modules, free modules, finitely presented modules and presentation matrices. Smith normal form of a matrix over a PID.

5. Structure theory: finitely generated modules over a PID, with an emphasis on finitely generated abelian groups. Subgroups of finitely generated free abelian groups. Applications to linear algebra: cyclic modules over F[t] (F a field); Jordan and rational canonical forms of matrices and their computation.

References: Artin's *Algebra* (second edition), Chapters 11, 12 and 14. Dummit–Foote's *Abstract Algebra* has good treatments of Gröbner bases (Section 9.6), and Jordan and rational canonical forms (Chapter 12).

Fields and Galois Theory.

1. Basic Theory: algebraic and transcendental elements. Degree of a field extension. Adjoining roots of polynomials. The primitive element theorem.

2. Finite fields: existence of a field \F^q of cardinality q = p^r, p prime. The elements of \F^q are the roots of x_q-x; uniqueness of \F^q up to isomorphism. The group \F_q^× is cyclic. Subfields of finite fields.

3. Isomorphisms of field extensions: finite Galois groups and Galois extensions, splitting fields. Action of the Galois group on the roots of a polynomial that splits completely in a Galois extension.

4. The Main Theorem of Galois Theory (i.e., the inclusion reversing correspondence between intermediate fields of a finite Galois extension and subgroups of its Galois group).

5. The Galois group of a polynomial: Galois groups of quadratic, cubic, quartic and cyclotomic polynomials. Solvability in radicals.

References: Artin's *Algebra* (second edition), Chapters 15 and 16.

Advanced Algebra.

1. Further ideal theory: nilradical, Jacobson radical, radical of an ideal, ideal quotients and the annihilator of an ideal, extension and contraction of ideals.

2. Further module theory: direct sums and products, direct and inverse limits; universal mapping properties. Complexes and exact sequences.

3. Multilinear Algebra: tensor, exterior and symmetric algebras over rings, with an emphasis on fields. Free, flat and projective modules.

4. Local rings and localization of modules: exactness of localization and local properties of modules (e.g., being zero or flat).

5. Chain conditions: Noetherian and Artinian modules and rings.

6. Finite generation: integrality, normal domains, and the going-up theorem.

References: Atiyah–Macdonald's *Introduction to Commutative Algebra*, Chapters 1, 2, 3, 5 (pp. 59–63), 6, 7 (pp. 80–82), 8. Reid's *Undergraduate Commutative Algebra* has a particularly good treatment of the Nullstellensatz and affine algebraic geometry (Chapters 4 and 5). Dummit–Foote's *Abstract Algebra* has a lot of great examples and exercises.