RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2020

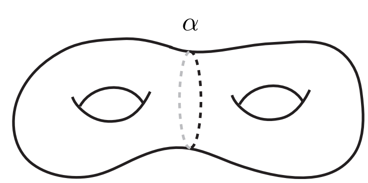
This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam. Upload to Gradescope (under Math 999) when you are finished.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let X be a path connected topological space. Under what conditions will two path classes γ and γ' from $x \in X$ to $y \in X$ give rise to the same isomorphism of $\pi_1(X, x)$ to $\pi_1(X, y)$. You should not just state a condition but should also prove that under the condition, that the isomorphisms are the same.

2. Let G be a finitely generated abelian group. Find a finite-dimensional path-connected topological space X_G with $\pi_1(X_G) = G$. You may use the fact that G will be isomorphic to a group of the form $\mathbb{Z}^k \oplus \mathbb{Z}_{q_1} \oplus ... \oplus \mathbb{Z}_{q_n}$ where $k \geq 0$ and the numbers $q_1, ..., q_n$ are powers of (not necessarily distinct) prime numbers. Show your work.

3. Let A be the surface of genus 2 show below and let α be the dashed curved ("the waist curve").



   Let B be another surface of genus 2 with waist curve β. Let X be $A \cup_f B$ identifying the two curves α and β (by a homeomorphism $f : \beta \to \alpha$).
   (a) Give a presentation for $\pi_1(X)$.
   (b) Calculate $H_p(X)$ for all p.

4. Let M be a closed, connected, oriented n-manifold with $\pi_1(M)$ finite and $\pi_1(M) \neq 1$.
   (a) Define the degree of a map between closed, connected, oriented n–manifolds.
   (b) Prove that there is no degree 1 map $f : S^n \to M$ where S^n is the n-sphere.
   (c) Show that there is a degree 1 map from M to S^n. You should explain how to construct an explicit map.

   Note: This also holds for manifolds M with non-trivial fundamendal group but it takes a little more work. For partial credit, you can do the problem for surfaces ($n = 2$) but you cannot assume that $\pi_1(M)$ is finite.

5. Suppose M is a compact, connected, orientable 3-dimensional manifold with non-empty boundary ∂M. If $\pi_1(M)$ is finite, prove that ∂M is a disjoint union of 2-spheres. (Hint: Calculate $H_1(\partial M; \mathbb{Q})$.)

6. Show that if M and N are smooth manifolds and if $p \in M$ and $q \in N$, then there is a canonical isomorphism
$$T_{(p,q)}(M \times N) = T_pM \oplus T_qN.$$
Describe this isomorphism in terms of (a) derivations and (b) linear combinations of partial derivatives with respect to coordinate charts.