RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2019

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ as indicated in the picture below. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

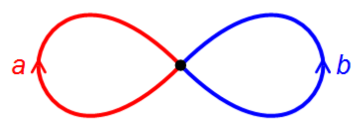

Figure 1. Wedge of two circles

2. Let G be a finitely generated abelian group. Describe a connected CW complex X_G with $\pi_1(X_G) \cong G$ and prove it has the correct fundamental group. You may use the fact that G will be isomorphic to a group of the form $\mathbb{Z}^k \oplus \mathbb{Z}_{q_1} \oplus ... \oplus \mathbb{Z}_{q_n}$ where $k \geq 0$ and the numbers $q_1, ..., q_n$ are powers of (not necessarily distinct) prime numbers.

3. Let $f : S^1 \times D^2 \to S^3$ be a *topological embedding*. That is, f is a homeomorphism of $S^1 \times D^2$ onto its image $f(S^1 \times D^2)$. The image $K = f(S^1 \times \{0\})$ is called a *(tame) knot* in S^3 . Use Mayer-Vietoris to compute $H_p(S^3 \setminus K))$. Hint: Think of $S^3 = \mathbb{R} \cup \{\infty\}$, as the one point compactification of $\mathbb{R}^3$.

4. Suppose M is a compact, connected, orientable 3-dimensional manifold with non-empty boundary ∂M. If $\pi_1(M)$ is finite, prove that ∂M is a disjoint union of 2-spheres. (Hint: Calculate $H_1(\partial M; \mathbb{Q})$.)

5. Prove that any continuous map $f : S^4 \to S^2 \times S^2$ has degree 0.

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined by
$$F(x, y) = (e^y \cos x, e^y \sin x, e^{-y}).$$
   (a) For which positive numbers r is F transverse to the 2-sphere of radius r, $S_r(0) \subset \mathbb{R}^3$?
   (b) For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?